Ancient and Mediaeval Astronomical Tables:

mathematical structure and parameter values

Astronomische tabellen uit oudheid en middeleeuwen:
wiskundige struktuur en parameterwaarden
(met een samenvatting in het Nederlands)

Proefschrift

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Contents

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Introduction

Since ancient times man has made systematic observations of the heavens and the heavenly bodies and has tried to find methods for predicting the positions of sun, moon and planets and the times of celestial phenomena like solar and lunar eclipses. From the fifth century B.C. onwards Babylonian astronomers used arithmetical schemes to predict the positions of the heavenly bodies by means of additions and subtractions. In the fourth century B.C. Greek astronomers began developing geometrical models for the motions of the heavenly bodies. These models were geocentric and were based on uniform motions along circles. Claudius Ptolemy, who made a large number of observations in Alexandria around the year 150 A.D., was the first astronomer to design a complete set of planetary models by means of which the positions of the sun, the moon and the five known planets could be predicted with an accuracy equal to that of the observations at the time, which were made by naked eye. Ptolemy computed sets of tables by means of which the planetary positions could be accurately predicted by only a small number of additions and multiplications. In the *Almagest* Ptolemy described his models extensively and explained the computation and use of his tables.[1]

Although Ptolemy's work was commented upon by later Greek scientists such as Pappus and Theon of Alexandria, there was no substantial further development in astronomy until the end of the eighth century. After the rapid expansion of Islam over an enormous territory ranging from India in the East to Spain in the West the Abbasid caliphs in Baghdad started to actively support science. Initially works of Persian and Indian origin were studied, but soon Greek manuscripts of important mathematical and astronomical works by classical Greek authors were collected on a large scale. These works were translated into Arabic and soon started to exert a large influence on the emergent Islamic science.[2] Islamic astronomers found that Ptolemy's tables no longer yielded accurate planetary positions. This was due to the fact that some values of astronomical parameters had changed in the course of time, and others had been determined by Ptolemy with insufficient accuracy. Thus Islamic astronomers started to make extensive new observations and compiled new sets of tables for determining the planetary positions. Between the eighth and the fifteenth centuries close to 200 different astronomical handbooks with tables and

[1] More information about Ptolemy can be found in the *DSB*-article by G.J. Toomer. The Greek text of the *Almagest* was published in Heiberg 1898–1903; an English translation with commentary occurs in Toomer 1984. For extensive descriptions of Ptolemy's planetary theory the reader is referred to Neugebauer 1957, pp. 191–207; Neugebauer 1975, vol. 1, pp. 21–261 or Pedersen 1974.

[2] When I speak of "Islamic", I am usually referring to Islamic culture, not to the religion. When I speak of "Arabic", I am referring to the Arabic language.

explanatory text, known by the name of *zīj*, were compiled.[3] In some cases these zījes merely contained a rearrangement of tables and explanations, but in many cases we find newly-determined values of astronomical parameters and tables computed to a higher degree of accuracy. Occasionally sophisticated modifications were made to the Ptolemaic planetary models.

From the tenth century onwards there was astronomical activity in Spain, then part of the Islamic world. Several Arabic astronomical works, among which were zījes, were translated into Latin and astronomical handbooks modelled on zījes were written in Spanish and Hebrew. Thus Islamic astronomy was crucial in the transmission of exact science to western Europe.

Astronomy played a key role in the Islamic exact sciences in general and in the development of mathematics in particular. Astronomy was the main field of application of advanced geometry, spherical trigonometry and numerical approximations. Therefore the investigation of zījes is an important aspect of the study of the history of Islamic science. In order to document the progress of Islamic astronomy we are particularly interested in the following questions: What are the relations between the many zījes that were written by Islamic authors? Which astronomers made observations of their own? Which astronomers introduced new values of astronomical parameters and used new methods of computation for the tables in their zījes? Below we will see that usually the answers to these questions cannot be found explicitly in the manuscripts of extant zījes, but have to be reconstructed from the numerical values in many separate tables. In this thesis I will present various mathematical methods that can be used to perform such reconstructions.

In Islamic zījes we generally find the following types of tables, most of which display non-trivial mathematical functions.[4]

- Chronological tables that can be used to convert the dates in the various calendars that were in use in the Middle Ages, in particular the Islamic lunar calendar, the Byzantine calendar and the Persian calendar. Usually the chronological tables include a list of kings or caliphs.
- Trigonometrical tables for the sine, versed sine, tangent and cotangent.
- Spherical astronomical tables that can be used to perform calculations connected with timekeeping: determination of the length of the day in equal hours or in seasonal hours, determination of the solar altitude, of rising times of arcs of the ecliptic, etc.
- Tables for the mean motions, the "equations" and the latitudes of the planets, by means of which planetary positions can be determined.
- Tables for the calculation of solar and lunar eclipses.
- Astrological tables for various functions, such as the horoscope, the astrological houses and the astrological aspects.
- Geographical tables displaying the longitudes and latitudes of various localities.

[3] The Arabic word *zīj*, pronounced as *zeedj*, is a Persian loanword that indicates a type of grid.

[4] An extensive description of functions occurring in Islamic zījes can be found in Kennedy 1956a, pp. 139–145. Section 1.3 lists the types of tables that are discussed in this thesis.

Of the more than 100 Islamic zījes that are extant very few have been published or studied extensively.[5] In the manuscripts of zījes that have come down to us we often find tables and explanatory text compiled by various astronomers. For example, in the only extant manuscript of the ninth-century *Mumtaḥan Zīj* we find a revision of the original zīj which contains material both from the original author and from various later astronomers.[6] In this thesis we will study the so-called *Baghdādī Zīj* (written in the year 1285), which has come down to us in its original form but turns out to be a compilation of rules and tables by various important early Islamic astronomers.[7]

Very often the compilers of zījes do not specify the origin of the tables they include. In such cases relations between tables in different zījes can be investigated by simply comparing all tabular values, but this is a cumbersome method that does not allow us to find relations between tables of different types. A more useful method of studying the relations between tables in zījes is to determine and compare mathematical properties of the tables. Since we can assume that the tables computed by a particular astronomer have certain mathematical properties in common, e.g. the computational technique, the values of the underlying parameters, or the use of certain auxiliary tables, it is possible to attribute tables on the basis of such properties and hence to determine the relations between various zījes.

In most zījes we do not find explicit information about the mathematical structure of the tables. Usually the explanatory text describes only how a particular table should be used. Even if the calculation of the tabular values is explained, we often find many differences between the table and a recomputation performed in accordance with the method indicated. In some cases values of the underlying parameters are mentioned, either in the explanatory text or in the headings of the tables, but these parameter values do not always agree with the values actually used for the computation. In exceptional cases two different values for a particular parameter are used in a single table.[8]

From the above it follows that, in order to investigate the relation between tables in zījes, it is very useful to have numerical and statistical methods at our disposal by means of which the mathematical structure and underlying parameter values of an astronomical table can be determined from the tabular values only. Until recently such methods have not been applied in a systematic way. Both E.S. Kennedy and D.A. King recomputed many tables from Islamic zījes, but they often based their recomputations on information found in the explanatory text and used only elementary mathematical methods for the analysis of the tables. R. Billard and R.P. Mercier made an extensive use of least squares estimation in order to establish the date and place of compilation of mean motion

[5]Kennedy 1956a lists the 125 Islamic zījes that were known in 1956. Since that time another 100 zījes have come to light. Examples of published zījes are al-Battānī's *Ṣābi' Zīj* (see Nallino 1899–1907), the Andalusian recension of al-Khwārizmī's zīj (Suter 1914), the zīj of al-Zarqālī (Millas Vallicrosa 1943–1950), al-Bīrūnī's *Masudic Canon* (al-Bīrūnī) and the *Mumtaḥan Zīj* (Yaḥyā ibn Abī Manṣūr). Extensive studies on zījes are Neugebauer 1962 (al-Khwārizmī), Toomer 1968 (*Toledan Tables*) and Debarnot 1987 (Ḥabash al-Ḥāsib). A large number of short studies on zījes and on astronomical tables in general can be found in Kennedy's collected works (*SIES*) and in the collected articles by King (*IMA*).

[6]See the manuscript Escorial árabe 927 or the facsimile edition Yaḥyā ibn Abī Manṣūr.

[7]See Chapter 4 of this thesis.

[8]An example of an oblique ascension table based on two different values of the obliquity of the ecliptic is mentioned in Section 2.6.1.

tables. Recently G.R. Van Brummelen described statistical tests for three mathematical properties of astronomical tables and applied these tests to the tables in Ptolemy's *Almagest*.

In this thesis I present a variety of numerical and statistical methods for determining the mathematical structure of ancient and mediaeval astronomical tables. In chapter 2 I will describe four statistical estimators that can be used to determine the parameter values for which a given table was computed. I will determine the accuracy of these estimators and will give extensive examples of their use. Note that the estimators require that certain conditions regarding the errors in the table to be analysed are satisfied. If this is not the case, incorrect conclusions concerning the values of the underlying parameters can be drawn. In case studies about the equation of time (Chapter 3) and the Baghdādī Zīj (Chapter 4) I will use various "ad hoc methods" for extracting further information about the method of computation of different types of tables. We will see that the combined use of parameter estimations and ad hoc methods enables us to find information about the method of computation of the investigated tables which can be used to draw conclusions about the origin of these tables.

The methods presented in this thesis are independent of the culture of which the astronomers under consideration formed part and independent of the language in which they wrote their works. Thus the methods can be applied equally well to tables in Greek, Arabic, Persian, Spanish, Latin and Hebrew manuscripts, and can for instance be extended to Indian astronomy. It follows that the methods are also very suitable for studying the transmission of astronomical knowledge from one culture to another.

It is not possible to give a general recipe for the analysis of ancient and mediaeval astronomical tables. The order in which statistical estimators and ad hoc methods should be applied depends on the mathematical properties of a particular table. In some cases we will first tackle a table by means of ad hoc methods (e.g. by checking the symmetry of the table or by recomputing the table for historically plausible parameter values) and will thus obtain an idea about the method according to which the table was computed. This idea can be verified by means of statistical methods. In other cases we will simply apply the parameter estimations at once and will draw conclusions from the patterns in the differences between the table and a recomputation for the estimated parameter values. On the basis of these conclusions we may be able to correct scribal errors or we can verify a possible method of computation (e.g. involving some type of interpolation) directly.

For certain tables the use of advanced statistical methods may seem somewhat exaggerated. In fact, there are many tables for which the underlying parameter values can be determined in a straightforward way. Examples are tables in which the parameter value occurs as one of the tabular values (e.g. in tables for the solar declination) and tables for which the parameter values agree with the information in the heading of the table or in the explanatory text (e.g. the tables from the Baghdādī Zīj analysed in Chapter 4).

However, there are two reasons why, even for such tables, the use of advanced statistical methods can be very useful. Firstly, there is a large category of tables for which the underlying parameter values cannot reliably be determined without the use of statistical estimators. This category includes, for example, tables that have very few sexagesimal places. For such tables two different values of an underlying parameter may lead to almost identical recomputations. For instance, in the so-called *Sanjufīnī Zīj*, compiled in Tibet in the 14th century, we find for the obliquity of the ecliptic both the values 23;32,30 and 23;35, which are very difficult to distinguish in right ascension tables with values calculated to an accuracy of minutes (see Section 2.6.1). Also for tables involving multiple unknown parameters and for tables containing many large errors in the tabular values the use of statistical estimators is mostly necessary in order to determine the underlying parameter values.

The second reason for using advanced statistical methods is that such methods do not only aid in determining the underlying parameter values of astronomical tables but they also assist in finding the precise tabulated function and the method of computation. Information concerning the method of computation of astronomical tables is interesting in itself, since our knowledge of the way in which ancient and mediaeval astronomers performed their extensive calculations is as yet very limited. Furthermore, determination of the method of computation of a table can be very useful in cases where the underlying parameter values do not give sufficiently detailed information about the origin of the table. For instance, the value 23;35 for the obliquity of the ecliptic is so common that no conclusions about the origin of a table can be drawn on the basis of this parameter value only. As explained above, if more information about the method of computation of the table is available, for instance if we know whether approximative methods or interpolation were used, we may be able to attribute tables whose authors could otherwise not have been identified.

In this thesis we find various illustrative examples of tables the origin of which could only be determined from information about the method of computation. For instance, both the table for the true solar longitude in the zīj of Kushyār ibn Labbān (see Section 2.6.3) and the table for the equation of time in Ptolemy's *Handy Tables* turn out to be computed by means of approximating methods and for parameter values that could not be expected on the basis of available historical information. Only the use of a least squares estimation for the multiple unknown parameters of both tables and the use of Fourier coefficients for determining properties of the function tabulated in Kushyār's table have made it possible to determine the precise method of computation and the values of the underlying parameters. Furthermore, in the chapter of this thesis concerning the *Baghdādī Zīj* we will see that the table for the equation of daylight in this zīj was computed from one of the other tables in the zīj by means of inverse linear interpolation in a sine table with accurate values for every 15 minutes of the argument (see Section 4.3.10). By combining this result with various others, we can conclude that several of the tables in the Baghdādī Zīj were copied from important earlier works that are otherwise no longer available to us.

When performing mathematical analyses of ancient and mediaeval astronomical tables, we must always bear in mind that the authors of such tables used methods of computation different from the methods we use nowadays. In some cases the Greek or mediaeval way of computing the values for a certain function is equivalent to the modern way in the sense that basically the same operations are performed in the same order. In other cases the Greek and mediaeval way of computing and the modern way are essentially different. This could imply for instance that the values of a particular table were calculated from auxiliary tables different from those we would expect on the basis of the modern formula (see for an example the definition of the right ascension in Section 4.3.8). Also essentially different types of errors may have been made during the calculation of tabular values. For example, the rounding of intermediate results of the calculation may have more influence on the accuracy of the tabular values than we expect.

Note that no problems occur as long as the method of computation used by an ancient or mediaeval astronomer leads to tabular values that are accurate to all places displayed. In that case we can use the modern formulae for the function concerned both for the estimation of the underlying parameters and for the recomputation of the table. In fact, there will be no way of discovering which method of computation was used by the author of the table. As soon as we find significant differences between a given ancient or mediaeval table and a modern recomputation, we must be aware of the possibility that a different method of computation might have been used. Usually the errors in the table can then be analysed in more detail in order to find which method of computation was used. After this method has been established, it may be necessary to perform the estimation of the underlying parameter values and the recomputation of the table anew.

Chapters 2 to 4 of this thesis can be read independently. Chapter 1 contains preliminaries to which the reader may refer while reading the remaining chapters. These preliminaries include: explanations of the terminology used throughout the thesis; extensive information about the types of errors that we find in astronomical tables; information about functions that occur in ancient and mediaeval astronomical handbooks; a description of the computer programs that I developed for the research presented in this thesis.

In Chapter 2 four statistical estimators are described that can be used to estimate the values of parameters in tables in zījes. In the introduction to this chapter (Section 2.1) the estimators are explained in an informal way. In the subsequent sections technical descriptions of the estimators are given and the accuracy of the estimators is computed. In Section 2.6 three examples of applications of the estimators are presented that can be read independently of the technical explanations.

In Chapter 3 the above-mentioned estimators for unknown parameter values and various other mathematical methods for analysing astronomical tables are demonstrated in four case studies of tables for the equation of time. The equation of time is a complicated function that involves four different parameters and was tabulated in two different ways. In Chapter 4 most of the trigonometric and spherical astronomical tables in the Baghdādī Zīj are analysed. The underlying parameter values of these tables present very few surprises, but many interesting details of the methods of computation of the tables will be revealed.

In the three examples of the application of parameter estimators in Section 2.6, in the case study of tables for the equation of time in Chapter 3 and in the case study of the trigonometric and spherical astronomical tables in the Baghdādī Zīj in Chapter 4 it will be shown very clearly that the mathematical methods for the analysis of astronomical tables presented in this thesis yield important information about the underlying parameter values and the method of computation. Often this information helps to determine the origin of the tables.

Chapter 1

Preliminaries

1.1 Terminology and Notation

Throughout this thesis I use various concepts that are related to astronomical tables, to the recomputation of such tables and to the errors in such tables. These concepts require precise definitions, which will be given below.

1.1.1 Numbers

The tabular values of practically all astronomical tables in ancient and mediaeval sources are displayed in *sexagesimal notation*. In transcribing sexagesimal numbers we will follow the convention that sexagesimal digits are separated by a comma and that the sexagesimal point is indicated by a semicolon. Thus the sexagesimal number 2,14;9,51 denotes $2 \cdot 60^1 + 14 \cdot 60^0 + 9 \cdot 60^{-1} + 51 \cdot 60^{-2}$. Usually the integer part of sexagesimal numbers was written in decimal form; thus the above number was written as 134;9,51. In certain applications the integer part was expressed in zodiacal signs and degrees, where a zodiacal sign corresponds to 30 degrees. We will indicate the use of zodiacal signs by a superscript "s" and thus obtain $4^{s}14;9,51$ for the above number.[1] Especially for values of parameters and for statistical data concerning tabular errors I will also use the modern notation of degrees, minutes, seconds, thirds, etc. Thus the latitude of Baghdad will be given as $33°25'$, the expected standard deviation of the rounding errors in an astronomical table with values to seconds as $17'''19^{\mathrm{iv}}$ (17 thirds, 19 fourths; cf. Section 1.2.4).

In astronomical handbooks in Arabic and Persian sexagesimal numbers were mostly written in the so-called *abjad* notation. This notation was based on the Greek way of denoting numbers, the letters of the Greek alphabet being replaced by Arabic letters. Every number from 0 to 9, every multiple of 10 from 10 to 90 and every multiple of 100 was denoted by a single Arabic letter (see Table 1.1). All other numbers were expressed by combining these letters. For instance, 25 was written as كه (ه + ك), 749 was written as ذمط (ط + م + ذ). Note that some regional variants of the notation given in table 1.1 were in use.

[1]In text this number would be written as "14 degrees 9 minutes 51 seconds of Leo" (note that Leo is the *fifth* zodiacal sign).

0	ō				
1	ا	10	ی	100	ق
2	ب	20	ك	200	ر
3	ج	30	ل	300	ش
4	د	40	م	400	ت
5	ه	50	ن	500	ث
6	و	60	س	600	خ
7	ز	70	ع	700	ذ
8	ح	80	ف	800	ض
9	ط	90	ص	900	ظ
				1000	غ

Table 1.1: The *abjad* notation of sexagesimal numbers

1.1.2 Rounding

In general, a *rounding* procedure will be indicated by the symbol r_k or the symbol r_u. The symbol r_k denotes a rounding to k sexagesimal fractional digits, r_u a rounding to unit u. In most cases the intended type of rounding is the so-called *modern rounding*: sexagesimal digits 30 and higher are rounded upwards, digits 29 and lower are rounded downwards. "Modern rounding" is rather an unfortunate term, since the majority of the ancient and mediaeval tables analysed in this thesis appear to be based on modern rounding. In some cases, however, the sexagesimal digit 30 was rounded downwards instead of upwards. Other methods of rounding, in particular *truncation* (all digits rounded downwards) and *upward rounding* (all digits rounded upwards), will be considered only incidentally.

1.1.3 Tables

Throughout this thesis a *table* from a zīj (astronomical handbook) will be indicated by the letter T. From the context it will always be clear which table is intended; usually T is the table that is being investigated in the section concerned. The tabular values of the table T will be referred to as $T(x)$, where x is the *argument* or *independent variable* of the table. The set of all arguments of a particular table will be indicated by a calligraphic letter, for instance $\mathcal{X}$. The *argument increment* is the difference between consecutive arguments. For all tables discussed in this thesis the argument is the length of a circle arc measured in degrees (astronomers from Antiquity and the Middle Ages thought in terms of circle arcs, where the complete circle corresponds to 360°, rather than in terms of angles). To calculate the accuracy of the statistical estimators introduced in Chapter 2, we will need the derivatives of the trigonometric functions. The use of degrees instead of radians leads to the appearance of factors $\pi/180$ in these derivatives.

The *unit* of a table is the greatest common divisor of all tabular values. For instance, if a table has been computed to an accuracy of seconds, the unit is 0;0,1. If, as a result of the method of computation, the seconds' place of all tabular values contains a multiple of 4, the unit is 0;0,4. This situation may arise when a table is computed by dividing whole numbers of minutes by 15. When I speak of *a table to minutes* or *tabular values to minutes* I mean that the tabular values were calculated to an accuracy of minutes.

The *tabulated function* will always be indicated by f_θ, where θ is either a single parameter or a parameter vector. For every argument x, $f_\theta(x)$ is called the *exact* or *precise* functional value. In order to obtain a tabular value for argument x the exact functional value must be rounded to the appropriate number of sexagesimal fractional digits, i.e. we have $T(x) = r_k(f_\theta(x))$. The values of the parameters that were used for the computation of a particular table will be referred to as the *underlying parameter values*. If a particular table was computed from one or more other tables, these tables will be referred to as the *underlying tables*.

By the *first quadrant* of a table I will mean the tabular values for arguments from 0° to 90°, the *second quadrant* being those for arguments from 90° to 180°, etc. A table will be called *symmetrical* whenever the tabular values in some of the quadrants can be calculated directly from those in other quadrants. For instance, the solar declination δ, which is defined by $\delta(\lambda) = \arcsin(\sin\lambda \cdot \sin\varepsilon)$ for every λ, satisfies the *symmetry relations* $\delta(180-\lambda) = \delta(\lambda)$ and $\delta(180+\lambda) = -\delta(\lambda)$ for every λ. The right ascension, in the first quadrant defined by $\alpha(\lambda) = \arctan(\tan\lambda \cdot \cos\varepsilon)$, satisfies the symmetry relations $\alpha(180-\lambda) = 180 - \alpha(\lambda)$ and $\alpha(180+\lambda) = 180 + \alpha(\lambda)$. The tabular values that can be computed directly from a given tabular value will be called "the symmetrical values" for that given tabular value.

Tabular values for functions like the planetary equations will be called *additive* or *subtractive* in order to indicate that they must be added or subtracted from a certain quantity. For instance, in order to calculate the true solar position, the correction called "solar equation" must in certain situations be added to the mean solar position, and in other situations must be subtracted from that position. Astronomers from Antiquity and the Middle Ages had to use the terms additive and subtractive since they did not use negative numbers. By adding a constant some astronomers made their corrections *always additive*, which means that the correction had to be added to a certain quantity in every situation.

A table will be said to have *double* or *quadruple entries* if every tabular value serves two or four values of the argument respectively. For instance, tables for the solar equation $\bar{q}$ as a function of the mean anomaly $\bar{a}$ usually have double entries, since we have $\bar{q}(360-\bar{a}) = -\bar{q}(\bar{a})$ for every $\bar{a}$. Depending on the entry that is used in a certain situation, the tabular value must be added or subtracted from the mean anomaly (cf. the previous paragraph). Tables for the sine often have quadruple entries since we have $\sin(180-x) = \sin x$ and $\sin(180+x) = -\sin x$ for every x.

Instead of saying "to look up a value in a table (for argument x)" I will usually say "to enter a table (with argument x)". This expression is a literal translation of the Arabic *dakhala jadwalan bi ...*.

1.1.4 Recomputations and Errors

If T is assumed to be a table for a function f_θ of which the tabular values are rounded according to the rounding procedure r_u, then a *recomputation* of T is a table having the same set of arguments as T and values $r_u(f_\theta(x))$ for every argument x. Thus T and its recomputation are given to the same number of sexagesimal fractional digits or, more generally, have the same unit. If no further indication is given, it is assumed that the modern formula for the tabulated function and the modern rounding procedure are used for the recomputation. If alternative methods of computation are used, which may for instance involve truncation, rounding during intermediate steps of the calculation or (inverse) interpolation, this will always be mentioned explicitly.

An *error in a table* or *error with respect to a recomputation* is the difference between a tabular value and a recomputed value. Thus, if we assume that the table T was computed for the function f_θ and that the tabular values were rounded according to the rounding procedure r_u, then the error $E(x)$ in the tabular value $T(x)$ is defined by $E(x) = T(x) - r_u(f_\theta(x))$. Consequently, the errors in a table have the same unit as the tabular values themselves. If $E(x)$ equals zero, the tabular value $T(x)$ will be said to be *correct*; otherwise, $T(x)$ is said to be *in error*. Errors that are extremely large compared to the general error pattern of a given table will be called *outliers*. Mostly outliers are caused by scribal errors, sometimes by computational mistakes (cf. Sections 1.2.1 and 1.2.3).

The *tabular error* $e_\theta(x)$ for argument x is the difference between the tabular value $T(x)$ and the exact functional value $f_\theta(x)$, i.e. $e_\theta(x) = T(x) - f_\theta(x)$. Since every tabular value involves at least a rounding error, tabular errors will practically always be non-zero (exceptions can for instance be found in sine tables for arguments 30° and 90°). The concept of a tabular error will only be used in statistical contexts, in particular when we want to calculate the accuracy of the parameter estimations described in Chapter 2 and when we want to check the conditions that must be satisfied for these estimations to be valid. From a plot of the tabular errors of a particular table one can often draw useful conclusions about the method of computation. An example of this is the plot of the tabular errors of an oblique ascension table computed by means of linear interpolation (see Figure 4.4). An extensive discussion of the probability distributions and the dependence of tabular errors is presented in Section 1.2.4.

When tabular errors are regarded as functions of the unknown parameter θ, they will be called *residuals*. In this thesis this occurs mainly in connection with least squares estimation, which determines the parameter θ in such a way that the sum of the squares of the residuals is minimized.

A *reconstructed* table is a historical table of which the tabular values could be determined exactly from an extant table that was computed from it. We can assume that the compiler of the extant table had an exact copy of the reconstructed table at his disposal. Examples of reconstructed tables can be found in Sections 2.6.3 and 4.3.14 of this thesis.

In various cases I will find an approximation for a table underlying a given astronomical table by using mathematical properties of the tabulated function. I will call the approximation an *extracted table*. Occasionally it can be seen that the values of an extracted table must be equal to the actual values of the underlying table. In such cases the extracted table will be called a reconstructed table (see above). In this thesis examples can be found of right ascension and equation of daylight tables extracted from a table for the oblique ascension (see for instance Sections 2.6.1 and 4.3.13) and of right ascension and solar equation tables extracted from a table for the equation of time as a function of the true solar longitude (see Sections 3.1.3 and 3.2.3).

1.1.5 Interpolation

It can be shown that several of the astronomical tables analysed in this thesis were computed with the aid of some type of *interpolation*. Thus only some of the tabular values were computed according to a (more or less) precise algorithm for the function concerned; the intermediate tabular values were determined by approximating the function between the precisely calculated values. The precisely calculated values will be called *nodes*, the intermediate tabular values, *internodal values*. From descriptions found in Greek and Islamic sources we know that interpolation was usually carried out by means of operations on tabular differences rather than by actually calculating the approximating function.[2] *Linear interpolation* was thus performed by distributing the difference between every two consecutive nodes among the intermediate values in such a way that the resulting tabular differences differed by one unit at most. In the case of *exact linear interpolation* the differences were distributed "evenly" between every two nodes. The result was equal to what we would obtain by calculating the values on the straight line between each two nodes and rounding appropriately. In the case of so-called *distributed linear interpolation* the difference between each two nodes was distributed in such a way that the resulting tabular differences were increasing or decreasing over as long as possible stretches of the argument. Thus, in modern terms, where the derivative of the tabulated function is increasing, the smaller differences between every two nodes preceded the larger ones and vice versa. Note that the error in values calculated by means of linear interpolation is generally largest half-way between two nodes.

Examples. Assume that the difference between two consecutive nodes is 8 units and that four intermediate tabular values must be filled in by means of linear interpolation. We obtain five intermediate tabular differences, namely three differences of 2 units and two differences of 1 unit. In the case of exact linear interpolation the differences are evenly distributed between the two nodes and their order becomes $2, 1, 2, 1, 2$. In the case of distributed linear interpolation the order depends on the local behaviour of the function: if the derivative of the function is increasing the order is $1, 1, 2, 2, 2$, if the derivative of the function is decreasing the order is $2, 2, 2, 1, 1$.

In order to calculate values of the inverse trigonometric functions, Greek and Islamic astronomers usually applied *inverse interpolation* in tables for the trigonometric functions.

[2] See for instance Mogenet & Tihon 1985, pp. 162, 170 and 254–264 for Theon's description of the special type of linear interpolation frequently used in Ptolemy's *Handy Tables* (this type will here be called "distributed linear interpolation" and is described below). See furthermore Kennedy 1962 and King 1973, pp. 354–357 for examples of descriptions of second-order interpolation schemes.

Thus the arcsine of a given number would be determined by approximating the sine between two consecutive given sine values and calculating the argument for which the approximation is equal to the given number. As in the case of ordinary interpolation, inverse interpolation was carried out by means of operations on tabular differences.[3]

1.2 Tabular errors

In the previous section a *tabular error* $e_\theta(x)$ for argument x was defined as the difference between the tabular value $T(x)$ and the precise value $f_\theta(x)$ of the tabulated function, i.e. $e_\theta(x) \stackrel{\text{def}}{=} T(x) - f_\theta(x)$. In this section we will study tabular errors in more detail. We will distinguish three components of a tabular error, namely (in the order in which they occur) the *computational error*, the *rounding error* and the *scribal error*. Each of these components will be discussed separately. Finally, I will discuss why tabular errors can be considered to be random variables and I will conjecture that in certain situations tabular errors can be considered to be mutually independent and uniformly distributed.

1.2.1 Computational errors

By *computational errors* I mean in the first place the relatively small errors in tabular values which result from rounding at intermediate steps of the calculation, from the use of possibly inaccurate underlying tables, from the use of (inverse) interpolation in other tables, etc. If the computational errors in a particular table are significantly smaller than the unit of the table, then the rounding of the calculated values to the unit of the table will generally make the errors imperceptible. If the computational errors are of the order of magnitude of the unit of the table or larger, they will lead to errors in the table. Most of the tables that I have investigated contain at least a couple of computational errors.

If the calculation of the values for a certain function is long and complicated, if intermediate results are rounded in the modern way (not truncated or rounded upwards) and if the calculation does not involve approximative methods that yield biased errors (such as linear interpolation), then it is reasonable to assume that computational errors have approximately a normal distribution with zero mean (cf. Section 1.2.4 below). In practice this assumption may not hold, particularly, in the case of tables that involve linear interpolation in one of the final steps of the computation. Such interpolation usually leads to groups of consecutive errors that have the same sign. Consequently, the errors cannot be considered to represent a sample from a distribution with mean zero and they will not be independent either. Note that, depending on the properties of the tabulated function and of the underlying tables, computational errors in a single table may have essentially different variances.[4]

By computational errors I also mean errors that result from mistakes during the calculation. Usually such mistakes lead to outliers, large errors that do not fit in the overall

[3] For a description of a second order inverse interpolation scheme, see Schoy 1927, pp. 41–42.

[4] An example of a type of table for which the variance of the computational errors often cannot be assumed to be a constant are tangent tables; cf. Section 4.3.3.

(0)	ع	⟷	ه	(5)	(4)	د	⟷	و	(6)	(1u)	ﺪ	⟷	ﻟ	(3u)
(2)	ب	⟷	د	(4)	(4)	د	⟷	ر	(7)	(1u)	ﺪ	⟷	ﻧ	(5u)
(2)	ب	⟷	ر	(7)	(6)	و	⟷	ر	(7)	(3u)	ﻟ	⟷	ﻧ	(5u)
(3)	ﺣ	⟷	د	(4)	(7)	ر	⟷	ن	(50)	(4u)	ﻤ	⟷	ﻧ	(5u)
(3)	ﺣ	⟷	ح	(8)	(9)	ط	⟷	ك	(20)	(40)	م	⟷	مر	(47)

Table 1.2: Common scribal errors

error pattern of the table. In exceptional cases it may be possible to find the origin of a computational error of this type by reconstructing the calculation of the tabular value step by step (it may be useful to start from the erroneous tabular value and to reconstruct the calculation in reverse order). Usually we will simply disregard computational errors of this type and we will perform parameter estimations using only the remaining tabular values.

1.2.2 Rounding errors

Rounding errors are the errors that result from the rounding of calculated functional values to the number of sexagesimal places (or, more accurately, to the unit) of the table to be computed.[5] Note that if the calculations have been performed with sufficient accuracy (i.e. if the computational errors described above are small enough), then the rounding errors are practically equal to the errors obtained by rounding the exact functional values to the number of sexagesimal places of the table. In that case the properties of the tabular errors can be determined by investigating the errors made in the rounding of the exact functional values. In Section 1.2.4 it will be argued that rounding errors can often be considered to be random variables with an approximately uniform probability distribution.

1.2.3 Scribal errors

Scribal errors are errors that occur during the copying of tabular values. The important Islamic zījes were copied many times, mostly by copyists who were not aware of the meaning of the text and the numbers they were copying. Consequently, large numbers of mistakes were made and the more often a particular zīj had been copied, the less reliable did its tabular values become.

Most scribal errors are the result of the confusion of Arabic numerals written in *abjad* notation (cf. Table 1.1 on page 8). A list of some of the most common confusions can be found in Table 1.2. Here every arrow between two abjad numerals indicates that these numerals can easily be confused. Scribal errors that occur particularly often are those of the forms $d_1u \leftrightarrow d_2u$, where d_1 and d_2 denote two different multiples of ten and u denotes

[5] The various types of rounding that were used in the computation of ancient and mediaeval astronomical tables are described in Section 1.1.2.

a particular number of units. In Arabic these numbers are written with two different so-called "initial forms" of the letters indicating multiples of ten. The initial forms that are most easily confused are listed in the third column of Table 1.2.[6] Specific examples are لس↔لب , نا↔با , نر ↔ لر and نط↔مط . Note that some of the confusions of this type are less plausible because of the special way in which certain combinations of Arabic letters are written (so-called ligatures), e.g. لا . On the other hand, the confusions نر↔ير ($1u \leftrightarrow 5u$) occur very often since the only difference between these two initial forms is the diacritical dot. From the table we can see that various other abjad numerals can only be distinguished by their diacritical dots. However, in many cases confusion is unlikely since the numerals concerned occur at different positions within a number. Thus confusion of ت (400) and ث (500) is plausible, but confusion of ز (7) and ر (200) is unlikely: in a number such as مز (47) the ز can only denote 7, in رم (240) the ر can only denote 200.

Note that there are significant differences in the way in which abjad numerals are written in different hands and in different regions and periods. For an overview of such differences the reader is referred to the article about Arabic numerical forms by Irani.[7] Depending on the hand in which the table to be copied is written, certain scribal errors are more probable than others. For instance, my experience is that in most manuscripts the abjad numerals ج (3) and د (4) can be easily distinguished, but in certain texts these numerals look almost identical.

A type of scribal error that also occurs regularly in Islamic astronomical tables is a shift of parts of a column. Copyists generally copied numerical tables column by column, thus for a particular column of arguments they would first copy the degrees, then the minutes, and then the seconds of the tabular values. If the copyist forgot a digit for a particular argument, all following digits would be shifted upwards by one line and the resulting tabular values would contain a digit that belonged to the following tabular value. Very often the copyist only discovered his mistake close to the end of the column when he found that the digit in the last line of the original would end up in the penultimate line of the copy. Instead of correcting all values, he usually inserted an arbitrary digit in order to fill up the empty position.

Sometimes we find two equal digits in the same place in two consecutive tabular values; in one case the digit is correct, in the other case it is clearly wrong. In such situations I assume that the copyist copied the same digit twice either by mistake or because one of the original digits was illegible. Only in exceptional situations can scribal errors of this type be corrected. Finally, it is my impression that scribal errors of ± 1 in any sexagesimal place occur relatively often. In many tables I found such scribal errors that could not be explained from a similarity of abjad numerals, but could be confirmed by using mathematical properties (see below) or by comparing the tabular values with copies of the same table in other manuscripts.

Both computational and scribal errors may be very large compared to the general error pattern of an astronomical table. However, unlike computational errors, scribal errors can very often be corrected. One possible way of doing this is to compare all available

[6] In most cases the yah, occurring in ى (10) and in ى (1u), was written without diacritical dots.

[7] Irani 1955.

manuscripts that contain a copy of the table under consideration. In this way we hope to restore the table as it *occurred in the original zīj*. If only one copy of a given table is at hand, we can use the mathematical properties of the tabulated function in order to correct the scribal errors. In Section 4.1.4 various methods of correction are explained in detail: inspection of the symmetry of the table; comparison of the tabular values with a preliminary recomputation; inspection of finite order tabular differences (note that the third of these methods includes the use of irregularities in the interpolation pattern for correcting scribal errors). By applying these mathematical methods we hope to restore the table as it *was originally computed*. This is precisely what we are interested in if we want to determine the underlying parameters of the table or its mathematical structure.

1.2.4 Distribution of tabular errors

In order to apply statistical estimators to numerical data in astronomical tables, it must be possible to consider the tabular values or, equivalently, the tabular errors as random variables. In principle the calculation of tabular values is a deterministic process. However, we can hope that the tabular errors *behave* like random variables in the sense that the conditions imposed by the statistical estimators that we want to apply can be assumed to be satisfied.

We will see that tabular errors and particularly rounding errors are in certain respects similar to numbers produced by random number generators. A random number generator is a deterministic device that produces a sequence of seemingly independent, uniformly distributed random numbers. The "randomness" and "independence" of these numbers come from the fact that once the numbers are given one cannot recognize the method that was used to calculate them. For example, no patterns will be visible in plots of the numbers and statistical tests will not reject the hypothesis that the numbers are independent or have a uniform distribution. We can hope that errors in tables for astronomical functions show the same type of randomness. Thus we expect that in certain situations it will be impossible to recognize patterns in the errors that we make by rounding exact functional values to the number of sexagesimal fractional digits of the table to be computed (the probability distributions of these rounding errors will be discussed below). Furthermore, the rounding at intermediate steps of the calculation, the use of rounded and possibly inaccurate values from other tables and the possible use of approximative methods can also be expected to lead to unpredictable differences between the calculated values and the exact functional values. Thus it seems reasonable to consider tabular errors as random variables.

If we want to test the randomness of tabular errors or if we want to perform a Monte Carlo analysis for a statistical estimator applied to astronomical tables, then the randomness must be introduced explicitly in the calculation of the tabular values. This can be done in various ways:

1. By considering the argument of the table as a random variable. We can for instance let the argument assume uniformly distributed values on the domain of the tabulated function.

2. By considering the underlying parameters of the table as random variables. We can for instance let the underlying parameters assume uniformly distributed values on historically plausible intervals.
3. By adding a random error to every calculated functional value. For example, in order to simulate rounding errors we can add uniformly distributed random variables, in order to simulate computational errors we can add normally distributed random variables.

Below we will see that the first two methods enable us to show that the distribution of rounding errors is approximately uniform. In practice we will prefer the second method to the first since we deal with tabular values for a fixed set of arguments instead of for randomly chosen ones. Often we will also prefer the second method to the third, since the second method makes it possible to study the dependence of tabular errors which occurs for instance if the number of sexagesimal fractional digits is small. If we use the third method, we must introduce any dependence between the errors ourselves. All Monte Carlo analyses carried out in this thesis were performed on the basis of the second method.

To apply the estimators for unknown parameter values described in Chapter 2 it is usually necessary for the tabular errors to be independent and to have mean zero and common variance. For the following types of tables these conditions may not hold:

- Tables that were computed by means of linear interpolation (cf. Section 1.1.5). In such tables we find small groups of errors of equal sign between accurately calculated values. The errors will not satisfy any of the three conditions mentioned above. If a table was computed by means of linear interpolation, we may disregard the interpolated values and may use only the accurately calculated tabular values for statistical purposes.
- Tables of functions that are (almost) linear. For such tables the rounding errors can still be uniformly distributed, but they cannot be assumed to be independent.
- Tables with small steps between consecutive arguments. For such tables the rounding errors are likely to be dependent.
- Tables with values calculated to very few sexagesimal places. Such tables often show dependence of consecutive rounding errors. This is illustrated in Figure 1.1, which displays the rounding errors of a solar equation table with values to minutes.[8] The dependence of the errors can clearly be seen in the regions where they are connected by line segments.
- Tables for functions like the tangent. The errors in a tangent table computed from a specific sine table increase rapidly as the argument approaches 90° (cf. Section 4.3.3). This implies that the tabular errors cannot be assumed to have a common variance.

In practice we always have to test whether the tabular errors of a particular table satisfy the conditions of a parameter estimation.[9]

[8]The tabulated function is $q(a) \stackrel{\text{def}}{=} \arcsin(e \cdot \sin a/60)$, where the argument a is the true solar anomaly and the solar eccentricity e is equal to the historical value 2;4,45. More information concerning the solar equation can be found in Section 1.3.

[9]In Knuth 1973–1981, vol. 2, pp. 38–113 elementary tests are described that can be used for testing

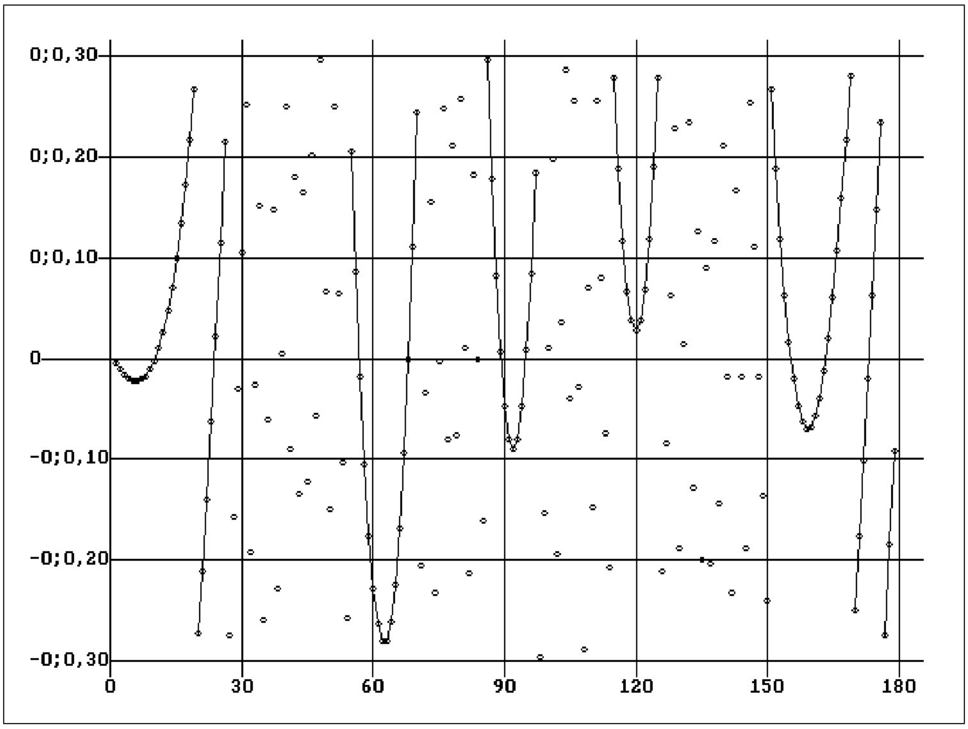

Figure 1.1: Rounding errors of a solar equation table with values to minutes

Distribution of rounding errors. I will now argue that under certain conditions the rounding errors in a correct table of an astronomical function have approximately a uniform distribution and can be assumed to be independent. First we note that the values produced by the linear congruential random number generator $x_{k+1} = (ax_k + c) \bmod m$ are rounding errors of a table with equidistant arguments and unit m for an exponential function $f(x) = ba^x + d$, where the constants b and d depend on a, c and the initial value x_0 of the sequence of random numbers.[10] It seems reasonable to expect that under certain conditions, in particular if the tabulated function is not (almost) linear and if the number of sexagesimal fractional digits of the tabular values is sufficiently large, the rounding errors in tables occurring in ancient and mediaeval astronomical handbooks have approximately a uniform probability distribution and are independent.[11] Thus we can for instance conjecture the following:

Let $T_{u,n}$ be a correct table with unit u for the non-linear function f, such that $T_{u,n}$ has n tabular values for equidistant arguments x_k, $k = 1, 2, 3, \ldots, n$ in a fixed interval. For every argument x_k the tabular error $e_{u,n}(x_k)$ defined by $e_{u,n}(x_k) = T_{u,n}(x_k) - f(x_k)$ is the rounding error that we make by rounding the exact functional value $f(x)$ to the unit of the table. I will assume that this rounding is performed in the modern way. Let

whether random variables are independent and have a uniform distribution. More information can be found in the literature on residual analysis; see for instance Draper & Smith 1981, Chapter 3. Methods for dealing with dependence between tabular errors are described in Seber & Wild 1989, Chapter 6.

[10] For a discussion of linear congruential random number generators see for instance Knuth 1973–1981, vol. 2, pp. 9–24.

[11] Cf. Problem 295 in *Statistica Neerlandica* **47** (1993), p. 88.

$F_{u,n}$ be the experimental distribution of the normalized rounding errors of the table $T_{u,n}$, i.e. $F_{u,n}(y)$ is the fraction of rounding errors smaller than uy for every y. Note that we have $F_{u,n}(y) = 0$ for every $y \leq -\frac{1}{2}$ and $F_{u,n}(y) = 1$ for every $y \geq +\frac{1}{2}$. Let F_u be the limiting distribution of $F_{u,n}$ for $n \to \infty$.

Conjecture. F_u converges to the uniform distribution on $[-\frac{1}{2}, +\frac{1}{2}]$ as u tends to 0.

For certain types of functions it may be possible to prove the conjecture by means of the theory of *uniform distribution modulo one*. Let $\{x\}$ denote the fractional part of the real number x. For a given sequence x_k, $k = 1, 2, 3, \ldots$ of real numbers let $A_K(a, b)$ be the number of terms x_k, $1 \leq k \leq K$, for which $\{x_k\} \in [a, b\rangle$. The sequence x_k, $k = 1, 2, 3, \ldots$ is said to be uniformly distributed modulo one if

$$\lim_{K\to\infty} \frac{A_K(a,b)}{K} = b - a \quad \text{for every } 0 \leq a < b \leq 1. \tag{1.1}$$

Using the notation introduced above, our conjecture can now be stated as follows: the sequence $\frac{1}{2} + f(x_k)/u$, $k = 1, 2, 3, \ldots$ is asymptotically uniformly distributed modulo one for $u \to 0$. An extensive overview of the theory of uniform distribution modulo one can be found in Kuipers & Niederreiter 1974.

Less general results can be obtained if we introduce randomness in an explicit way as explained above. By means of an unpublished theorem by J.H.B. Kemperman it can be shown that for a randomly chosen argument the distribution of the rounding error converges to the uniform distribution as the unit of the table tends to 0 or, equivalently, as the number of sexagesimal fractional digits tends to infinity.[12] Since the roles of the argument and an underlying parameter can be interchanged, the same holds for a randomly chosen value of an underlying parameter. M.A. Stephens investigated the rounding errors of a sine table with values to two decimal places by considering the independent variable to be uniformly distributed on the interval $[0, \frac{1}{2}\pi]$. He found that the distribution of the rounding error is close to uniform on most of the interval $[-0.005, +0.005]$. However, the density of the rounding errors tends to infinity as the error approaches 0 from above. Stephens showed that one would need extremely large numbers of tabular values in order to distinguish the distribution of the rounding error from a uniform distribution.[13]

As a corollary of the assumed uniform distribution of rounding errors we expect that for sufficiently large k all digits from 0 to 59 occur approximately equally often in the k-th sexagesimal fractional place of the tabular values in a given table. This implies, for instance, that if most of the values in a particular table have a final sexagesimal digit which is a multiple of 4, we can assume that the table was computed by means of a method that leads to multiples of 4 in the final digit. If all final digits are equally probable, the probability that, for example, 80 or more of 90 final digits are multiples of 4 is only $2.3 \cdot 10^{-37}$. Incidental final digits that are not a multiple of 4 can usually be attributed to copying mistakes.[14]

[12] For a proof of this property the reader is referred to van Dalen 1989, pp. 118–119.
[13] See Stephens 1990.
[14] The property described here is used in Sections 3.1.3 and 4.3.14 of this thesis.

Dependence of rounding errors. We expect that the rounding errors in a table for a non-linear function will become independent as the unit of the table approaches 0 or, equivalently, as the number of sexagesimal fractional digits tends to infinity. For instance, we may conjecture that the joint experimental distribution of rounding errors for arguments with fixed distances converges to the joint distribution of independent uniform variables as the unit of the tabular values approaches 0.

1.3 Functions occurring in zījes

Throughout this thesis tables of many different types of functions are analysed. This section presents an alphabetical list of all these functions together with the symbols used to denote them, the modern formulae, and references to more extensive descriptions. In most cases these references are to sections of Chapter 4, in which tables from the Baghdādī Zīj for thirteen different functions are analysed. For each function the paragraph headed **definition** of the section of Chapter 4 to which reference is made gives the definition of the function, the modern formula according to which the function can be calculated, the underlying parameters, and mathematical properties of the function such as symmetry relations. Furthermore it is indicated which specific methods can be used to analyse tables for the function concerned, how the underlying parameters can be determined and what difficulties may occur when statistical estimators for the parameters are used. Functions for which no table from the Baghdādī Zīj is analysed in Chapter 4 are described more extensively in the remainder of this section.[15]

In the formulae for functions occurring in zījes we use the following symbols to denote the independent variables:

x	general independent variable
λ	true solar longitude
λ'	true solar longitude measured from the winter solstitial point
$\bar{\lambda}$	mean solar longitude (linear function of time)

The following symbols are used to denote the underlying parameters:

R	radius of the base circle for trigonometric functions
ε	obliquity of the ecliptic
ϕ	geographical latitude
e	solar eccentricity
λ_A	solar apogee
c	epoch constant of the equation of time
D	conversion factor of the equation of time

[15]Further information about all functions can be found in various books that discuss the Ptolemaic planetary theory, e.g. Neugebauer 1957, pp. 191–207; Neugebauer 1975, vol. 1, pp. 21–261 and Pedersen 1974. More specific information about functions that occur in Islamic zījes can be found in Kennedy 1956a, pp. 139–145.

The following is the alphabetical list of all functions. Note that some of the functions in the list are expressed in terms of other functions.

function	*formula*	*reference*
ascensional difference	*see:* equation of daylight	
cotangent	$\operatorname{Cot} x = R \cdot \cot x$	Section 4.3.3
declination	*see:* solar declination	
equation of daylight	$\Delta(\lambda) = \arcsin(\tan\delta(\lambda) \cdot \tan\phi)$	Section 4.3.10
equation of time	$E_h(\lambda) = \frac{1}{D}(\lambda + q(\lambda) - \alpha(\lambda) + c)$	Section 3.1.1
	$\bar{E}_h(\bar{\lambda}) = \frac{1}{D}(\bar{\lambda} - \alpha(\bar{\lambda} - \bar{q}(\bar{\lambda})))$	Section 3.1.1
hour length	$H(\lambda) = (90 + \Delta(\lambda))/6$	Section 4.3.12
length of daylight	$L(\lambda) = (90 + \Delta(\lambda))/7\frac{1}{2}$	Section 4.3.11
"method of declinations"	$q_\delta(\bar{\lambda}) = q_{\max} \cdot \dfrac{\delta(\bar{\lambda} - \lambda_A)}{\varepsilon}$	see below
normed right ascension	$\alpha'(\lambda') = \arctan(\tan\lambda'/\cos\varepsilon)$	see below
oblique ascension	$\rho(\lambda) = \alpha(\lambda) - \Delta(\lambda)$	Section 4.3.13
right ascension	$\alpha(\lambda) = \arctan(\tan\lambda \cdot \cos\varepsilon)$	Section 4.3.8
second declination	$\delta_2(\lambda) = \arctan(\sin\lambda \cdot \tan\varepsilon)$	Section 4.3.5
sine	$\operatorname{Sin} x = R \cdot \sin x$	Section 4.3.1
sine of the equation of daylight	$s_\Delta(\lambda) = R \cdot \tan\delta(\lambda) \cdot \tan\phi$	Section 4.3.9
solar altitude	$h_s(\lambda) = 90 - \phi + \delta(\lambda)$	Section 4.3.6
solar declination	$\delta(\lambda) = \arcsin(\sin\lambda \cdot \sin\varepsilon)$	Section 4.3.4
solar equation	$q(a) = \arcsin(\frac{e}{60}\sin a)$	see below
	$\bar{q}(\bar{a}) = \arctan\left(\dfrac{e\sin\bar{a}}{60 + e\cos\bar{a}}\right)$	see below
	$q(\lambda) = \arcsin(\frac{e}{60}\sin(\lambda - \lambda_A))$	see below
	$\bar{q}(\bar{\lambda}) = \arctan\left(\dfrac{e\sin(\bar{\lambda} - \lambda_A)}{60 + e\cos(\bar{\lambda} - \lambda_A)}\right)$	see below
tangent	$\operatorname{Tan} x = R\tan x$	Section 4.3.3
tangent of declination	$t_\delta(\lambda) = R \cdot \tan\delta(\lambda)$	Section 4.3.7
true solar longitude	$\lambda(\bar{\lambda}) = \bar{\lambda} - \bar{q}(\bar{\lambda})$	Section 2.6.3
versed sine	$\operatorname{Vers} x = R \cdot (1 - \cos x)$	Section 4.3.2

Solar Equation. In the Ptolemaic solar model the true position of the sun is determined by subtracting a variable correction, called the *solar equation*, from the mean solar position, which is a linear function of time.[16] The solar equation was usually tabulated as a function of the mean solar anomaly $\bar{a}$, which is defined as the difference between the mean solar longitude $\bar{\lambda}$ and the constant solar apogee denoted by λ_A: $\bar{a} = \bar{\lambda} - \lambda_A$. As a function of $\bar{a}$ the solar equation $\bar{q}$ is given by the modern formula

$$\bar{q}(\bar{a}) = \arctan\left(\frac{e \sin \bar{a}}{60 + e \cos \bar{a}}\right), \tag{1.2}$$

where the underlying parameter e is the solar eccentricity. Because of the symmetry relation $\bar{q}(360 - \bar{a}) = -\bar{q}(\bar{a})$, the solar equation as a function of the mean anomaly was usually tabulated with double entries (cf. Section 1.1.3). The maximum value $q_{\max}$ of the solar equation was often mentioned separately in explanatory text. The maximum satisfies the equation $q_{\max} = \arcsin(e/60)$ and is assumed approximately for arguments 92 and 268.

In early Islamic astronomical works various approximations for the solar equation were used, many of which were described in a treatise by al-Bīrūnī.[17] In Section 2.6.3 a table based on one such approximation, namely the so-called "method of declinations", will be analysed. This approximation, indicated by q_δ, can be computed from a declination table by a simple multiplication:

$$q_\delta(\bar{a}) = q_{\max} \cdot \frac{\delta(\bar{a})}{\varepsilon}, \tag{1.3}$$

where $q_{\max}$ is the maximum solar equation, ε is the obliquity of the ecliptic and the function $\delta(\bar{a}) \stackrel{\text{def}}{=} \arcsin(\sin \bar{a} \cdot \sin \varepsilon)$ is the "solar declination",[18] which also depends on the obliquity. The "method of declinations" satisfies the symmetry relations $q_\delta(180-\bar{a}) = q_\delta(\bar{a})$ (which is not satisfied by the solar equation $\bar{q}$ itself) and $q_\delta(360-\bar{a}) = -q_\delta(\bar{a})$, and can therefore be tabulated with quadruple entries. The maximum value of $q_\delta(\bar{a})$ is assumed for arguments $\bar{a} = 90$ and $\bar{a} = 270$. For the obliquity of the ecliptic in tables based on the "method of declinations" until now only the Ptolemaic value $\varepsilon = 23;51$ has been encountered. The "method of declinations" was applied in the zījes by al-Khwārizmī[19] and by Yaḥyā ibn Abī Manṣūr.[20]

Incidentally the solar equation was tabulated as a function of the true solar anomaly a, which is defined as the difference between the true solar longitude λ and the solar apogee λ_A: $a = \lambda - \lambda_A$. We have

$$q(a) = \arcsin(\tfrac{e}{60} \sin a), \tag{1.4}$$

[16] A detailed explanation of the Ptolemaic solar model can for instance be found in Pedersen 1974, pp. 122–158. See also Section 3.1.1 of this thesis. Note that the solar equation as defined here by means of a modern formula can assume negative values as well. Ancient and mediaeval astronomers did not use these negative values; instead of adding a negative solar equation they subtracted the absolute value of the equation.

[17] See Kennedy & Muruwwa 1958.

[18] Note that the quantity $\delta(\bar{a})$ is not the actual declination of the sun. The name "method of declinations" merely refers to the way in which the approximation for the solar equation is computed.

[19] See Suter 1914, pp. 132–137 and Neugebauer 1962, p. 95.

[20] See Kennedy 1977 and Section 2.6.3 of this thesis.

where q denotes the solar equation and e is again the solar eccentricity. The maximum solar equation $q_{\max}$ follows from $q_{\max} = \arcsin(e/60)$ and is assumed for arguments $a = 90$ and $a = 270$. The solar equation as a function of the true solar anomaly satisfies the symmetry relations $q(180 - a) = q(a)$ and $q(180 + a) = -q(a)$ for every a.

Normed Right Ascension. The normed right ascension $\alpha'(\lambda')$ can be defined as follows. Let $\lambda' \in [0, 360]$ be given and let Λ be the point on the ecliptic which is such that the arc between the winter solstitial point and Λ, measured in the direction of the solar motion, has length λ'. Let P be the orthogonal projection of Λ onto the equator. Then the normed right ascension $\alpha'(\lambda')$ of λ' is the length of the arc between the orthogonal projection onto the equator of the winter solstitial point and P, again measured in the direction of the solar motion. For $\lambda' \in [0, 90\rangle$ the normed right ascension can be found from the modern formula

$$\alpha'(\lambda') = \arctan(\tan\lambda' / \cos\varepsilon), \tag{1.5}$$

where ε denotes the obliquity of the ecliptic. For $\lambda' \in [90, 360]$ the normed right ascension can be determined by means of the symmetry relations $\alpha'(180 - \lambda') = 180 - \alpha'(\lambda')$ and $\alpha'(180 + \lambda') = 180 + \alpha'(\lambda')$, which hold for every λ'.

Note that the only difference between the ordinary right ascension defined in Section 4.3.8 and the normed right ascension is the point from which the arcs of the equator and the ecliptic are measured. In the case of the ordinary right ascension both arcs are measured from the vernal point; in the case of the normed right ascension the ecliptical arc is measured from the winter solstitial point, the equatorial arc from the orthogonal projection of the winter solstitial point onto the equator. If α denotes the ordinary right ascension, we have $\alpha'(\lambda') = \alpha(270° + \lambda') - 270°$ for $\lambda' \in [0, 90]$ and $\alpha'(\lambda') = \alpha(\lambda' - 90°) + 90°$ for $\lambda' \in [90, 360]$. The normed right ascension is particularly convenient for determining the longitude of the ascendant.[21]

1.4 Computer Programs

Practically all astronomical tables in ancient and mediaeval sources display values in sexagesimal notation (see Section 1.1.1). Consequently, it seemed useful to write, as part of my doctoral research, one or more computer programs that can handle tables with such values. As far as I know, no convenient software existed that could be used both for editing, printing and saving tables in sexagesimal notation, for performing statistical estimations of the underlying parameter values of such tables, for applying various "ad hoc methods" for determining the mathematical structure of the tables and, finally, for recomputing them. In fact, I do not know of any program that can handle large sets of sexagesimal numbers in a flexible way. Statistical tests of the properties of tabular errors can easily be carried out in most statistical software packages once the tabular errors have been converted to decimal notation. On the other hand, the parameter estimations

[21] See Neugebauer 1975, vol. 1, p. 42.

explained in Chapter 2 would require a lot of extra programming if performed in a standard statistical package. Thus it seemed worth while to combine all the above-mentioned functions into one large program, to which I gave the name Table-Analysis.

1.4.1 Description of Table-Analysis

My program Table-Analysis (abbreviated to *TA*) can be used on any MS-DOS compatible personal computer.[22] In order to speed up extensive calculations such as parameter estimations, the program makes use of a mathematical coprocessor whenever one is present. Depending on the amount of memory available, *TA* can handle from 20 to 30 astronomical tables at once. Each of these tables contains at most 400 tabular values, which are all accurate to 10 sexagesimal places and can have 5 sexagesimal fractional places. Every tabular value consists of a *manuscript entry* and a *corrected entry.* The manuscript entry will generally contain the tabular value as found in the manuscript. If, during the analysis of the table concerned, a scribal or other type of error is discovered, a corrected value can be stored in the corrected entry. Usually the manuscript entries will be used for a critical edition of the table, the corrected entries for parameter estimation and other mathematical operations.

Since entering tabular values from a manuscript into a computer program is the most unpleasant part of research on astronomical tables, *TA* includes a command that assists the user in this task. For many types of tables *TA* first asks for one, two or three tabular values for specific arguments, then calculates preliminary estimates for the underlying parameters from these values, and finally predicts the remaining tabular values. The user only needs to edit the predicted values, for instance by pressing the + or − key in order to increase or decrease the predicted value by one unit.

A large variety of operations can be performed on any table that has been entered in *TA*. First we want to save a newly entered table on disk. Any number of tables can be saved in files that have the extension .ZIJ. One such file may contain the tables from a particular zīj or from a particular manuscript of a zīj. Next we can print sets of tables in a number of different lay-outs, either on the screen or on a line-printer. We can also write sets of tables to an Ascii file. It is possible to make a graphical plot of one or more tables on the screen; in this way irregularities in a table can be spotted and patterns in the errors of a table can be recognized. Any two tables of the same type can easily be compared: *TA* gives the number of differences between the two tables and the mean and standard deviation of the differences.

A number of mathematical operations on astronomical tables are available in *TA*. For instance, one can check which tabular values satisfy the symmetry of the tabulated function and one can compute tables of finite order differences in order to recover scribal errors or investigate the use of interpolation. Furthermore, it is possible to "extract" the tables used for the computation of a particular oblique ascension table or a table for

[22] *TA* was programmed in Borland's Turbo Pascal 5.0. Version 1.0 was released in June 1992 and handles approximately 50 types of tables, mostly trigonometrical, spherical astronomical, solar and lunar tables. The size of the executable file of this version is 300 Kb.

the equation of time (see Section 1.1.4). If, in a certain case, the tabulated function is unknown, its Fourier coefficients can be approximated for the purpose of discovering more information about the function.

All four estimators for unknown parameter values described in Chapter 2 are incorporated in *TA*. A single unknown parameter can be estimated by means of the Weighted Estimator or the Least Number of Errors Criterion, a translation parameter by means of the Fourier Estimator, and multiple unknown parameters by means of the Least Squares estimator. For all estimators it is necessary to check whether the conditions regarding the tabular errors are satisfied. In *TA* this can be done in two ways, namely by carrying out a number of elementary tests for independence and / or uniformity of the tabular errors and by performing a Monte Carlo analysis for tables of the type concerned. If desired, the tabular errors can be written to an ASCII file in decimal notation and can be further tested with the aid of standard statistical software.

After the underlying parameters have been determined, we want to recompute the table under consideration. The ordinary command for this purpose makes use of modern formulae and applies no rounding at intermediate stages of the calculation (except for the machine rounding to approximately 18 decimal places). If the differences between table and recomputation are relatively large, we may try to find a better recomputation by staying closer to the methods of computation of ancient and mediaeval astronomers. To this end *TA* provides the possibility for computing a table according to a user-specified formula. This formula can involve rounding or truncation of the tabular values at any stage of the calculation. By means of two separate commands one can perform linear interpolation and inverse linear interpolation between the values of any table that has been entered in *TA*.

The research presented in this thesis was performed almost exclusively with the aid of the program TABLE-ANALYSIS. All figures except 3.1 were obtained from *TA* by means of the plot command described above and a shareware program that writes the contents of a graphical screen to a PostScript file. All tables in this thesis were obtained by means of a special option of the print command in *TA* which produces a LaTeX source file. In the course of my research I built up a database containing approximately 250 tables from 20 different manuscripts of zījes. Both the executable code of the program *TA* and the tables in this database can be purchased from the author.

1.4.2 Other Programs

In addition to TABLE-ANALYSIS I wrote three more computer programs that I used regularly during my research. Firstly, I wrote a modified version of *TA* which deals with mean motion tables only. Mean motion tables are linear tables displaying multiples of the mean motion of a planet per day for periods of months, years and groups of years. Since the underlying parameter values often have 5 or more significant sexagesimal fractional digits, the computation of mean motion tables (and hence also the analysis of such tables) requires more accuracy than is provided by *TA*. In my program MEAN-MOTION all arithmetic is performed sexagesimally with an accuracy to 15 places. Apart from that, the operations included are similar to those in the program *TA*. Since mean motion tables generally contain very few errors, the Least Number of Errors Criterion, explained in Section 2.5, turns out to be particularly useful. By applying this criterion, the underlying value for the mean motion per day can be determined much more accurately than by "squeezing" that value from the single mean motion value that covers the largest period.[23]

Secondly, in order to convert the dates in various calendars that were used in mediaeval zījes, I wrote a program CALH.[24] This program includes 21 versions of 9 different calendars, among which are the Julian, Gregorian, Islamic, Byzantine and Persian calendars. After a date in one of these calendars has been entered, the corresponding dates in all 9 calendars are displayed together with the day of the week and the Julian day-number. By pressing a single key, the user of CALH can switch between the different versions that were used of certain calendars, for instance the civil and astronomical versions of the Islamic calendar.

Finally, I wrote a sexagesimal calculator SCTR which functions like an ordinary pocket calculator.[25] SCTR performs all operations, including the square root and the trigonometric functions to an accuracy of 12 sexagesimal fractional places. The integer parts of the sexagesimal numbers can be displayed in three different ways: decimal (as in most astronomical tables in zījes); in complete circles, zodiacal signs and degrees; and purely sexagesimal. Up to 10 sexagesimal numbers can be stored in memory labelled with a description. In SCTR sexagesimal numbers can easily be converted to their decimal equivalents.

Both CALH and SCTR are available in memory-resident versions. These versions are first loaded into memory and can then at any time be run from within other programs (such as *TA* or a word processor).

[23] The mean motion per day can be approximated from every tabular value in a mean motion table by dividing the total mean motion in the period concerned by the length of that period expressed in days. In this way the tabular value that gives the mean motion in the largest period covered by the table will lead to the most accurate approximation for the mean motion per day.

[24] The H in CALH stands for "historical". I wrote another version of my calendar conversion program which is called CALM and includes various modern calendars, e.g. the Persian Hijra-Shamsi calendar.

[25] SCTR is based on the so-called Reverse Polish system. This implies that, in order to calculate, for example, $1;30 + \sin 45^\circ$, one would type `1;30 ⟨ENTER⟩ 45 SIN +`.

Chapter 2

Estimating Unknown Parameter Values

2.1 Introduction

Most of the tables in mediaeval Islamic astronomical handbooks, so-called zījes, are based on one or more astronomical parameters. Examples of such parameters are the radius of the base circle in trigonometric tables; the obliquity of the ecliptic and the geographical latitude in spherical astronomical tables; the mean planetary motion per day, the eccentricity, the apogee and the epicycle radius in planetary tables.[1] Since most Islamic zījes are based on the Ptolemaic planetary theory (or variations of this theory),[2] the tabulated functions are essentially the same in different handbooks. Thanks to Ptolemy's *Almagest* and the explanatory sections of many mediaeval zījes, we know the general algorithm according to which many types of tables were computed. Sometimes we also find descriptions of interpolation, alternative methods of computation and approximate methods.

The situation with respect to the underlying parameter values of tables in zījes is quite different. Over the centuries new values for the parameters were calculated from fresh observations. The values for some parameters, such as the obliquity of the ecliptic and the solar eccentricity and apogee, actually changed in the course of time. For other parameters, like the planetary mean motions, more accurate values could be calculated because data over longer stretches of time became available. For parameters like the geographical latitude, the values to be used depended on the location for which the table was intended. Table 2.1 shows a number of values for the obliquity of the ecliptic that can be found in astronomical handbooks in the Ptolemaic tradition. Although some of these values (in particular 23;51 and 23;35) were used in a large number of different zījes, others (e.g. 23;33 and 23;30,17) can be considered to be typical for certain astronomers. Note

[1] For more information about the functions and the astronomical parameters occurring in tables in zījes, see Neugebauer 1975, vol. 1, pp. 21–230; Pedersen 1974; and Kennedy 1956a, pp. 139–145. See also Section 1.3 of this thesis.

[2] The Ptolemaic planetary theory is explained extensively in Neugebauer 1975 and Pedersen 1974.

author	location	time (A.D.)	obliquity value
Ptolemy	Alexandria	150	23;51
	India	500	24; 0
Yaḥyā ibn Abī Manṣūr	Baghdad	830	23;33
al-Battānī	Raqqa (Syria)	900	23;35
al-Ṭūṣī	Maragha (Iran)	1250	23;31
Alfonso X	Sevilla	1270	23;32,29
al-Kāshī	Samarkand	1325	23;30
Ulugh Beg	Samarkand	1340	23;30,17

Table 2.1: Values for the obliquity of the ecliptic according to various astronomers

that the decrease in the historical values for the obliquity corresponds to the decrease in the actual value from 23;40,37 in the year 150 to 23;31,30 in 1340.

Many zījes were compilations of material from earlier works which could involve different values for one particular parameter. Even zījes that were originally consistent are sometimes only extant in recensions containing many later additions that may involve different values for the underlying parameters. In the numerous cases where the source of the material is not indicated, we can hope to derive information about the origin of the tables through the values of the underlying parameters. However, in many cases neither the headings of the tables nor the explanatory text mention these values explicitly. Even if information about the parameter values is presented, we cannot always rely on its correctness. We conclude that for establishing the parameter values on which a given astronomical table is based (and hence for determining the origin of that table) it is very useful to have reliable mathematical methods which make use only of the tabular values.

Up till now very few authors have used numerical methods to find the underlying parameter values of Islamic astronomical tables. E.S. Kennedy has succeeded in recomputing a number of tables from zījes by means of computer programs. He tried several historical values for the unknown parameters in order to obtain the best fit with the table. Successful examples of this approach can be found in his collected works.[3] For instance, Kennedy (together with Salam) showed that the maximum lunar latitude involved in one of the lunar tables in the sole surviving manuscript of the 9th century *Mumtaḥan Zīj*, which was compiled by a team of astronomers headed by Yaḥyā ibn Abī Manṣūr, is the same as the value attributed to him in later sources.[4] Furthermore, Kennedy showed that a solar equation table in the *Ashrafī Zīj*, attributed to Yaḥyā and computed by means of an approximate method involving the solar declination, is based on the Ptolemaic value of the obliquity of the ecliptic rather than on the Indian value or any of the Islamic values.[5] Kennedy started a parameter file, which now contains over 1500 values for many different

[3]Kennedy et. al. 1983.
[4]See Salam & Kennedy 1967, p. 496.
[5]See Kennedy 1977, p. 184 and Section 2.6.3 of this chapter, where a related table is analysed.

parameters. He also published a list of all geographical coordinates occurring in Islamic sources, arranged according to locality, source, latitude and longitude.[6]

D.A. King has made use of computers for recomputing a large number of Islamic astronomical tables. In some cases he verified that the parameter values mentioned in the headings and in the explanatory text actually underlie the tables, in other cases he determined the underlying parameters by means of trial-and-error. Furthermore he investigated the resulting error patterns. Examples of King's approach can be found in his papers on timekeeping and on lunar crescent visibility tables.[7]

R. Billard made use of the method of least squares in order to determine the epoch of Indian mean motion tables. He calculated the year for which the planetary positions obtained from the tables show the best agreement with the actual planetary positions as computed according to modern formulae.[8] R.P. Mercier extended Billard's method and estimated by means of least squares not only the epoch of the mean motion tables, but also the meridian of reference.[9]

On several occasions J.D. North gave useful methods for approximating unknown parameter values in astronomical tables. In his book "Richard of Wallingford" he presented numerous simple rules for determining the eccentricities and epicycle radii underlying planetary equation tables.[10] In "Horoscopes and History" North studied the problem of determining the method of computation of a given horoscope, as well as of finding the latitude for which the horoscope was intended. Again he indicated several methods for approximating unknown parameters from one or two tabular values. In addition, he determined a more accurate approximation for an unknown parameter by computing the arithmetic mean of approximations computed from single tabular values.[11]

J.P. Hogendijk described a method for determining the underlying parameters of a table for predicting lunar crescent visibility.[12] By assuming a value for the obliquity of the ecliptic, he computed values for the geographical latitude from pairs of tabular values. Only if the table was computed according to a so-called "solar criterion" are the values for the latitude obtained in this way expected to be equal.

After determining an approximation for an unknown parameter, North and Hogendijk search in its neighbourhood for historically plausible values, i.e. round numbers or parameter values which are attested in the sources. They do not study the errors in the approximations systematically.

The methods described above suffice to give *rough* values for unknown parameters, and in some cases leave little doubt about the historical values underlying the table under consideration. However, especially in the case of multiple unknown parameters, more sophisticated approximation methods are necessary to find reliable results (or any results at all). Such sophisticated methods may answer questions like:

[6] Kennedy & Kennedy 1987.
[7] See King 1973, King 1978 and King 1987.
[8] See Billard 1971, Chapter 2, pp. 41-68.
[9] See Mercier 1987 and Mercier 1989.
[10] See North 1976, vol. 3, pp. 192–195.
[11] See North 1986, pp. 11–16.
[12] See Hogendijk 1988a, p. 31.

- Which neighbouring parameter values yield an equally good or even better recomputation than the approximations found? In particular, are there parameter values not attested in the sources which yield a better recomputation than attested values close to the approximations?
- Which of two very close attested parameter values more probably underlies the table? Note, for instance, that two right ascension tables with values to minutes based on the attested obliquity values $23°32'30''$ and $23°33'$ may be very difficult to distinguish.[13] Furthermore a declination table with entries to minutes would not display the exact obliquity for 90° if it were based on a value to seconds.

In this chapter, statistical estimation theory will be used to find reliable approximations for single or multiple unknown parameter values in astronomical tables. The estimators that will be introduced make use of all tabular values (possibly disregarding values calculated by means of interpolation and a small number of so-called "outliers"). As a result, these estimators will be more accurate than approximations computed from only one or two tabular values and less sensitive to computational and scribal errors. Therefore the estimators can also distinguish between parameter values that lead to nearly identical tables.

Apart from approximations for the unknown parameter values, we need a measure of the accuracy of these approximations. Thus for three of the four estimators presented, so-called "confidence intervals" will be determined. These intervals have a fixed probability (usually 95 %) of containing the unknown parameters. By means of the confidence intervals it can be decided which historically plausible values for the unknown parameters underlie the table in question. If no known historical values are contained in the confidence intervals, we have a strong indication that the table is based on hitherto unattested values.

Now short informal descriptions will be given of the estimators presented in this Chapter. The discussion of the estimators in Sections 2.2 to 2.5 and the determination of their accuracy are highly technical. Therefore the reader without a background in statistics is advised to read the following short descriptions in order to obtain a general understanding of the estimators and then proceed with the examples in Section 2.6. In these examples, tables from three different manuscripts are analysed by means of the estimators described below combined with various "ad hoc methods". It will become clear that this combination is an extremely powerful tool for determining both the mathematical structure and the underlying parameter values of ancient and mediaeval astronomical tables. All estimators presented below are incorporated in my computer program TABLE-ANALYSIS described in Section 1.4.1.

1. Weighted Estimator. The weighted estimator can be used to determine a single unknown parameter value of an astronomical table. It extends a straightforward method of determining the parameter from a single tabular value in such a way that the information in all tabular values is used as efficiently as possible. As an example, assume that we

[13] If such tables were computed correctly, they would differ in only 4 out of 90 entries.

have a table for the right ascension $\alpha_\varepsilon(\lambda) = \arctan(\tan\lambda \cdot \cos\varepsilon)$, where the argument λ is the solar longitude and the parameter ε the obliquity of the ecliptic. From every tabular value $T(\lambda)$ we can estimate the obliquity by means of

$$\varepsilon \approx \arccos\left(\frac{\tan T(\lambda)}{\tan\lambda}\right). \tag{2.1}$$

For example, if T is a table with correct values to minutes based on obliquity $\varepsilon = 23;35$, we find from $T(45) = 42;30$ that $\varepsilon \approx 23;36,16$. In order to utilize the information contained in all tabular values, we can take the average of all approximations for ε obtained in this way. However, it turns out that the approximations obtained for arguments close to 0° or 90° are less accurate than those obtained for arguments around 45°. For instance, from the correct tabular value $T(89) = 88;55$ we find $\varepsilon \approx 22;37,20$ (!). To solve this problem, we calculate a so-called "weighted" average: inaccurate approximations to ε are multiplied by a small constant, so that they have less influence on the average.

2. Fourier Estimator. The Fourier estimator can be applied to tables of functions f of the form $f(x) = g(x - c)$, where c is an unknown constant and g is periodic and odd ($g(x + 360°) = g(x)$ and $g(x) = -g(-x)$ for every x). We call c a "translation parameter": a shift of c leads to a horizontal translation of the graphical representation of the function. An example is the solar equation, for which the longitude of the apogee is a translation parameter. By means of the Fourier estimator a translation parameter can be estimated even if the tabulated function or the values for the other underlying parameters are unknown. The estimator is named after Fourier because it makes use of the so-called "Fourier coefficients" for a periodic function. Van der Waerden used the Fourier estimator to determine the solar apogee from the lengths of the months given in Greek and Indian sources, but he did not determine the accuracy of the approximations that he found in this way.[14]

3. Least Squares Estimation. If a table has two or more unknown parameter values, we can make use of a least squares estimation to find approximations for these values. If tabular values $T(x)$ for a function f_θ are given, then a least squares estimate of the parameter vector θ is a vector $\widehat{\theta}$ which makes the sum $\sum_x (T(x) - f_\theta(x))^2$ of the squares of the tabular errors as small as possible. Only if f_θ is a linear function can the least squares estimates be calculated in a straightforward way. In other cases it is necessary to use an iterative optimization procedure as explained in Section 2.4. Many statistical software packages contain such a procedure. In my computer program TABLE-ANALYSIS described in Section 1.4.1, least squares estimations can easily be carried out for a large number of different types of tables.

4. Least Number of Errors Criterion. Since we expect the number of errors in an astronomical table to be reasonably small, it can be useful to determine for which parameter values the number of differences between a given table and a recomputation is minimized. This so-called "Least Number of Errors Criterion" has been applied by Kennedy and King to find out which of a number of attested parameter values most

[14]See Van der Waerden 1952 and Van der Waerden 1985.

probably had been used for the calculation of a particular table. For tables with a single unknown parameter a more systematic approach is possible, which gives us an interval containing all parameter values for which the number of differences between the table and a recomputation is as small as possible. For instance, for a correct right ascension table to minutes based on obliquity 23;35, it turns out that the number of differences between the table and a recomputation is zero whenever $\varepsilon \in \langle\, 23;34{,}56\,,\ 23;35{,}3\,]$, one if $\varepsilon \in \langle\, 23;34{,}39\,,\ 23;34{,}56\,] \cup \langle\, 23;35{,}3\,,\ 23;35{,}13\,]$. From the intervals found in this way, a historically plausible value for the unknown parameter can be chosen.

We have seen that we can use the weighted estimator and the least number of errors criterion to determine the value of a single unknown parameter in an astronomical table. For multiple unknown parameters, least squares estimation can be applied. For all three estimation methods it is necessary to know the type of function that has been tabulated. If the function is unknown, Fourier coefficients may yield useful information. By means of the Fourier estimator an unknown translation parameter can be estimated; in the example in Section 2.6.3 it will be shown that other information about the function can be obtained as well.

For all types of estimators certain conditions regarding the tabular errors, i.e. the differences between the tabular values and the functional values, must hold. In particular, all errors should be of the same order of magnitude, and the errors should be uncorrelated. After performing an estimation, it is always necessary to check whether these conditions are satisfied, for instance by recomputing the table for the found estimates for the unknown parameters. Whenever obvious patterns (for example sinusoidal) are visible in the tabular errors, the conditions are unlikely to hold and the obtained confidence intervals are not valid.

The **weighted estimator** is described extensively in Section 2.2.1. First the estimators based on a single tabular value are defined and their bias and variance calculated. Then approximately optimal weights are determined and the weighted estimator is defined as a weighted average of the separate estimators. Bias and variance of the weighted estimator are calculated and it is shown that the distribution of the estimator is approximately normal. Finally a 95 % confidence interval for the unknown parameter can be determined. Examples of the weighted estimator are given for the case of the obliquity of the ecliptic in a right ascension table (Section 2.2.2) and the eccentricity in a solar equation table (Section 2.2.3). In both cases the weighted estimator turns out to be asymptotically biased if we let the number of tabular values approach infinity. In most practical situations, however, the bias is negligible with respect to the standard deviation. In Remark 1 of Section 2.2.1 a different type of asymptotics is suggested which removes the asymptotic bias of the weighted estimator.

The **Fourier estimator** is discussed extensively in Section 2.3. First it is shown that under certain general conditions a translation parameter λ_A of a periodic function f satisfies $\tan \lambda_A = -a_1/b_1$, where a_1 and b_1 are the coefficients of $\cos x$ and $\sin x$ in the Fourier series of f. Next it is shown that unbiased estimates $\widehat{a}_1$ and $\widehat{b}_1$ of these Fourier coefficients can be determined, provided that equidistant tabular values for a complete period are available. The calculation of the variance of these estimates and of the bias and variance of the estimator $\widehat{\theta} \stackrel{\text{def}}{=} -\widehat{a}_1/\widehat{b}_1$ for $\tan\lambda_A$ is straightforward. In most situations the distribution of $\widehat{\theta}$ turns out to be approximately normal. In order to find an approximate 95 % confidence interval, the variance of the tabular errors must be estimated using a finite Fourier series with approximated coefficients (see Section 2.3.3). A number of special cases in which the tabular errors cannot be assumed to be independent is treated in Section 2.3.2. For instance, if the function f satisfies the symmetry $f(x) = -f(x + \frac{1}{2}P)$ for every x, where P is the period of f, the variance of $\widehat{\theta}$ is twice as large. If λ_A is equal to one of the arguments or lies precisely between two arguments, the estimator $\widehat{\theta}$ turns out to be degenerate.

In Section 2.4 the principle of **least squares estimation** is explained and two iterative methods for determining a least squares estimate are indicated. It is shown how a 95 % confidence region for the unknown parameter vector can be calculated, and a warning is given for functions with strongly correlated parameters like the equation of daylight. In Section 2.5 the **least number of errors criterion** as described above is formalized and a statistical justification for the criterion is suggested. The discussion of both the least squares estimation and the least number of errors criterion is very brief. As far as the least squares estimation is concerned, more extensive information on the accuracy of the estimates and the convergence of the iterative methods could be given. As far as the least number of errors criterion is concerned, extensive statistical investigations need to be done in order to exploit its possibilities in full.

2.2 The Weighted Estimator for a Single Unknown Parameter

In this section an estimator will be described that can be used for most types of tables with a single unknown parameter value. The estimator extends a straightforward method of determining the parameter value from a single tabular value and uses a weighted average to make the best possible use of the information in all tabular values. Bias and variance of the "weighted estimator" thus defined will be determined (Section 2.2.1) and examples will be given for the estimation of the obliquity of the ecliptic in a right ascension table (Section 2.2.2) and the eccentricity in a solar equation table (Section 2.2.3). Applications of the weighted estimator can be found in Sections 2.6.1 and 2.6.2.

2.2.1 General Theory

Let T be a given table for the function $f_\theta(x)$ with tabular values $T(x)$ and a single unknown parameter θ. The tabular errors $e_\theta(x)$ are defined as $e_\theta(x) \stackrel{\text{def}}{=} T(x) - f_\theta(x)$.[15] We assume that the errors are mutually independent and have distributions with common mean 0 and variance $\sigma^2 > 0$. This implies that all tabular values contain errors of the

[15] Tabular errors are discussed in Section 1.2. Note that, according to the definition of $e_\theta(x)$, also the rounding errors of a correctly computed table are called tabular errors.

same order of magnitude.[16] Suppose that there exists a function g: $\mathbb{R}^2 \to \mathbb{R}$ such that for all x and θ, $g(x, f_\theta(x)) = \theta$.[17] Then $\widehat{\theta}_x \stackrel{\text{def}}{=} g(x, T(x))$ is a reasonable estimator for the unknown parameter θ for every value of x. First we will compute the accuracy of these separate estimators. We will assume that g has first and second derivatives with respect to the second variable and write g' for $\frac{\partial}{\partial y} g$ and g'' for $\frac{\partial^2}{\partial y^2} g$.

By computing the mathematical expectation of the right-hand side of the Taylor series

$$\begin{aligned} \widehat{\theta}_x &= g(x, T(x)) \\ &= g(x, f_\theta(x) + e_\theta(x)) \\ &= g(x, f_\theta(x)) + e_\theta(x) \cdot g'(x, f_\theta(x)) \\ &\quad + \tfrac{1}{2} e_\theta^2(x) \cdot g''(x, f_\theta(x)) + \mathcal{O}(e_\theta^3(x)), \qquad (e_\theta(x) \to 0) \end{aligned} \tag{2.2}$$

we find that $\widehat{\theta}_x$ has approximately an expected value $E\widehat{\theta}_x = \theta + \frac{1}{2}\sigma^2 \cdot g''(x, f_\theta(x))$ and hence that its bias $b_{\widehat{\theta}_x}$ can be approximated by

$$\widehat{b}_{\widehat{\theta}_x} \stackrel{\text{def}}{=} \tfrac{1}{2}\sigma^2 \cdot g''(x, f_\theta(x)). \tag{2.3}$$

Furthermore, its variance $\operatorname{Var} \widehat{\theta}_x$ is approximately equal to

$$\widehat{S}^2_{\widehat{\theta}_x} \stackrel{\text{def}}{=} \sigma^2 \cdot {g'}^2(x, f_\theta(x)). \tag{2.4}$$

Note that whenever σ is small, the approximate bias $\widehat{b}_{\widehat{\theta}_x}$ is probably small compared to the approximate standard deviation $\widehat{S}_{\widehat{\theta}_x}$ of $\widehat{\theta}_x$.

N.B. Here and in the sequel when I use the phrase "the bias of the estimator $\widehat{\theta}$ is approximately $\widehat{b}_{\widehat{\theta}}$ and its variance $\widehat{S}_{\widehat{\theta}}$" I mean "$\widehat{\theta}$ has a probability distribution which can be well approximated by a distribution with bias $\widehat{b}_{\widehat{\theta}}$ and variance $\widehat{S}_{\widehat{\theta}}$".

If a particular tabular value $T(x)$ contains a scribal error, the estimate $\widehat{\theta}_x$ computed from it may be highly misleading. But even if no scribal error occurs, the most accurate separate estimator $\widehat{\theta}_x$ may be insufficient to determine the unknown parameter. For example, if one tabular value from a right ascension table to minutes is used to compute the underlying value of the obliquity of the ecliptic, the result is a range of possible obliquity values with a width of at least $4\frac{1}{2}$ minutes, which could easily contain various

[16] As is explained in Section 1.2.4 this is a reasonable assumption for many tables. Note that if a particular tabular value seems to contain no error at all, like the value $T(90) = 23;35,0,0$ in a declination table, we can normally use it to find the underlying parameter directly and there will be no need to apply a statistical estimator. In many tangent tables the size of the error increases towards argument 90, which implies that the tabular errors do not have a common variance. I have not seriously tried to develop the theory of the weighted estimator for the case where the tabular errors are dependent. If groups of tabular errors do not satisfy the conditions for the weighted estimator, the tabular values concerned may be disregarded. This approach was used in the examples of Sections 2.6.1 and 2.6.2, and can also be applied to tangent tables.

[17] In most cases such a function exists in an explicit closed form. An exception is the estimation of the obliquity ε from a table for the so-called "method of declinations" $q_\delta(\bar{\lambda}) = q_{\max} \cdot \arcsin(\sin(\bar{\lambda} - \lambda_A) \cdot \sin\varepsilon)/\varepsilon$.

attested values. Furthermore a single solar equation value to minutes would not allow us to distinguish between the historical values 2;4,35$\frac{1}{2}$ and 2;4,45 of the solar eccentricity (corresponding to maximum solar equations of $1°59'$ and $1°59'10''$ respectively).

A more accurate estimator $\widehat{\theta}$ for the unknown parameter θ can be obtained by computing a weighted average of the separate estimators $\widehat{\theta}_x$:

$$\widehat{\theta} = \frac{1}{W}\sum_x w_x \widehat{\theta}_x, \tag{2.5}$$

where w_x denotes the weights and $W = \sum_x w_x$. The weights will be chosen in such a way that less accurate separate estimators have a smaller influence on the average. In Appendix 2.A.1 it is shown that the variance of $\widehat{\theta}$ can be minimized by choosing the weights inversely proportional to the variances of the separate estimators $\widehat{\theta}_x$. Therefore, in the present situation the weights $w_x = \dfrac{1}{g'^2(x, f_\theta(x))}$ are approximately optimal.[18] If we choose the weights accordingly, I will refer to $\widehat{\theta}$ as "the weighted estimator for a single unknown parameter". Note that in practice we will have to use weights $w_x = \dfrac{1}{g'^2(x, f_{\theta_0}(x))}$, where θ_0 is a preliminary estimate of θ.[19] The variance $\operatorname{Var}\widehat{\theta}$ of the weighted estimator is approximately equal to

$$\frac{1}{W^2}\sum_x w_x^2 \widehat{S}^2_{\widehat{\theta}_x} \approx \frac{1}{W^2}\sum_x w_x \sigma^2 = \frac{\sigma^2}{W} \stackrel{\text{def}}{=} \widehat{S}^2_{\widehat{\theta}}, \tag{2.6}$$

where $W = \sum_x \frac{1}{g'^2(x, f_{\theta_0}(x))}$. The bias $b_{\widehat{\theta}}$ is approximately equal to

$$\frac{1}{W}\sum_x w_x \widehat{b}_{\widehat{\theta}_x} \approx \frac{1}{W}\sum_x \frac{\frac{1}{2}\sigma^2 g''(x, f_{\theta_0}(x))}{g'^2(x, f_{\theta_0}(x))} \stackrel{\text{def}}{=} \widehat{b}_{\widehat{\theta}}. \tag{2.7}$$

Using the inequality of Cauchy–Schwarz it follows that

$$\frac{\left|\widehat{b}_{\widehat{\theta}}\right|}{\widehat{S}_{\widehat{\theta}}} = \frac{\left|\sum_x \dfrac{\frac{1}{2}\sigma^2 g''(x, f_{\theta_0}(x))}{g'^2(x, f_{\theta_0}(x))}\right|}{\sigma\sqrt{\frac{1}{g'^2(x, f_{\theta_0}(x))}}} \le \tfrac{1}{2}\sigma\sqrt{\sum_x \left(\frac{g''(x, f_{\theta_0}(x))}{g'(x, f_{\theta_0}(x))}\right)^2}. \tag{2.8}$$

This gives us a possibility to check whether the bias of $\widehat{\theta}$ is negligible compared to its standard deviation. It can be noted that in many cases the upper bound for $|\widehat{b}_{\widehat{\theta}}|/\widehat{S}_{\widehat{\theta}}$ obtained in this way is not a sharp one. For the estimation of the obliquity of the ecliptic

[18]Remember that g' denotes the first derivative of g with respect to the second variable.

[19]If necessary, the weighted estimator can be calculated twice. The first time θ_0 can be taken equal to an estimate of the unknown parameter determined from a single tabular value or to a plausible historical value. The second time θ_0 can be taken equal to the outcome $\widehat{\theta}$ of the first weighted estimation.

in a right ascension table and of the eccentricity in a solar equation table I will compute an accurate approximation for $|\widehat{b}_{\widehat{\theta}}|/\widehat{S}_{\widehat{\theta}}$ by approximating sums with Riemann integrals.[20]

In Appendix 2.A.2 a version of the Central Limit Theorem for the case of bounded variances is presented, which can be found in Loève 1977–1978.[21] Using this theorem, a condition regarding the distributions of the random variables $w_x(\widehat{\theta}_x - E\widehat{\theta}_x)/\sqrt{W}$ can be given which is sufficient for the weighted estimator $\widehat{\theta}$ to converge to a normal distribution as the number of separate estimators $\widehat{\theta}_x$ tends to infinity. It will be shown in Appendix 2.A.2 that if the tabular errors $e_\theta(x)$ can be assumed to have "reasonable" distributions, in particular if they are uniformly bounded or, more generally, if they have uniformly bounded $2+\delta$ moments for some $\delta > 0$, then $\widehat{\theta}$ will have approximately a normal distribution with mean $\theta + \widehat{b}_{\widehat{\theta}}$ and variance $\widehat{S}^2_{\widehat{\theta}}$, provided that the number of separate estimators is sufficiently large. Consequently, an approximate 95 % confidence interval for the unknown parameter θ is given by

$$\left\langle \widehat{\theta} - \widehat{b}_{\widehat{\theta}} - 1.96\widehat{S}_{\widehat{\theta}}\ ,\ \widehat{\theta} - \widehat{b}_{\widehat{\theta}} + 1.96\widehat{S}_{\widehat{\theta}} \right\rangle . \tag{2.9}$$

For every table to which I apply the weighted estimator I will perform a Monte Carlo analysis to test the validity of the confidence interval thus obtained. Furthermore I will test the tabular errors for independence.

Remark 1. So far we have considered the variance σ^2 of the tabular errors $e_\theta(x)$ to be a constant. In practice, however, it is more realistic to assume that the variance is related to the total number of tabular values n. In the remainder of this section, whenever I take this relationship into account, I will denote the variance of the tabular errors by σ_n^2. It seems reasonable to assume that tables for a particular function with $60n$ tabular values are computed to roughly one sexagesimal place more than tables with n values, since then the differences between consecutive tabular values expressed in units of the final sexagesimal digit are approximately the same. In practice it seems that the errors in the tabular values expressed in units of the final sexagesimal digit become somewhat larger as the number of fractional digits increases, but it seems realistic to assume that at least the penultimate digit is reasonably accurate. Thus it follows that $\sigma_n^2 = \mathcal{O}(1/n^2)$ $(n \to \infty)$.

We will see in Sections 2.2.2 and 2.2.3 that if we take the relation between the number of tabular values and the variance of the tabular errors into account, the bias of the weighted estimator for both the obliquity of the ecliptic in a right ascension table and the eccentricity in a solar equation table can be shown to be asymptotically negligible compared to the standard deviation. Furthermore, the assumption that the number of sexagesimal fractional digits of the table increases with the number of tabular values, actually makes it possible to assume that the tabular errors are independent even if the number of tabular values is very large. A fixed number of sexagesimal fractional digits

[20] See Sections 2.2.2 and 2.2.3.
[21] See Loève 1977–1978, vol. 1, pp. 300–308.

would in practice lead to consecutive tabular values which are equal and therefore have dependent errors.

Remark 2. In many cases it turns out to be more convenient to estimate a transformed parameter $h(\theta)$ instead of θ itself. For instance, in the case of the right ascension $\alpha_\varepsilon(\lambda) = \arctan(\tan\lambda \cdot \cos\varepsilon)$, we will estimate $\cos\varepsilon$ instead of ε; in the case of the solar equation $q_e(\bar{a}) = \arctan\left(\dfrac{e\sin\bar{a}}{60 + e\cos\bar{a}}\right) = \arctan\left(\dfrac{\sin\bar{a}}{\frac{60}{e} + \cos\bar{a}}\right)$, we will estimate $\frac{60}{e}$ (see Sections 2.2.2 and 2.2.3). Thus we make use of separate estimators $\widehat{h(\theta)}_x \stackrel{\text{def}}{=} h(g(x, T(x)))$ instead of $\widehat{\theta}_x = g(x, T(x))$. Since $\frac{\partial}{\partial y}h(g(x,y)) = h'(g(x,y)) \cdot \frac{\partial}{\partial y}g(x,y)$, we have

$$\widehat{S}^2_{\widehat{h(\theta)}_x} = |h'(g(x, f_\theta(x)))|^2 \cdot \widehat{S}^2_{\widehat{\theta}_x} = |h'(\theta)|^2 \cdot \widehat{S}^2_{\widehat{\theta}_x}. \tag{2.10}$$

Furthermore, since $\frac{\partial^2}{\partial y^2}h(g(x,y)) = h'(g(x,y)) \cdot \frac{\partial^2}{\partial y^2}g(x,y) + h''(g(x,y)) \cdot \left(\frac{\partial}{\partial y}g(x,y)\right)^2$, it follows that

$$\widehat{b}_{\widehat{h(\theta)}_x} = h'(\theta) \cdot \widehat{b}_{\widehat{\theta}_x} + \tfrac{1}{2}h''(\theta) \cdot \widehat{S}^2_{\widehat{\theta}_x}. \tag{2.11}$$

We find that the approximate bias $b_{\widehat{h(\theta)}}$ of the weighted estimator $\widehat{h(\theta)}$ for $h(\theta)$ is given by $b_{\widehat{h(\theta)}} = h'(\theta) \cdot b_{\widehat{\theta}} + \frac{1}{2}nh''(\theta)\widehat{S}^2_{\widehat{\theta}}$, where n is the number of separate estimators $\widehat{h(\theta)}_x$. The approximate variance $\widehat{S}^2_{\widehat{h(\theta)}}$ of $\widehat{h(\theta)}$ satisfies $\widehat{S}^2_{\widehat{h(\theta)}} = |h'(\theta)|^2 \cdot \widehat{S}^2_{\widehat{\theta}}$. If $\langle a, b\rangle$ is an approximate 95 % confidence interval for $h(\theta)$, an approximate 95 % confidence interval for θ can simply be computed as $\langle h^{-1}(a), h^{-1}(b)\rangle$ (or $\langle h^{-1}(b), h^{-1}(a)\rangle$, if h is decreasing).

Remark 3. The weighted estimator as introduced above can also be applied to tables of functions based on more than one parameter, provided that all but one of the parameter values are known. For example, let $\delta_\varepsilon(\lambda) = \arcsin(\sin\lambda \cdot \sin\varepsilon)$ be the solar declination, dependent on the obliquity of the ecliptic ε, and let $\Delta_{\varepsilon,\phi}(\lambda) = \arcsin(\tan\delta_\varepsilon(\lambda) \cdot \tan\phi)$ be the equation of daylight. If we can assume a value for the obliquity, we can estimate $\theta \stackrel{\text{def}}{=} \tan\phi$ using the function $g(x,y) = \sin y / \tan\delta_\varepsilon(x)$. Thus our separate estimators $\widehat{\theta}_\lambda$ are given by

$$\widehat{\theta}_\lambda = \frac{\sin T(\lambda)}{\tan\delta_\varepsilon(\lambda)},$$

their approximate bias and variance by

$$\widehat{b}_{\widehat{\theta}_\lambda} = -\frac{1}{2}\frac{\pi^2\sigma^2\theta}{180^2} \qquad \text{and} \qquad \widehat{S}^2_{\widehat{\theta}_\lambda} = \frac{\pi^2\sigma^2}{180^2}\left(\frac{1}{\tan^2\delta_\varepsilon(\lambda)} - \theta^2\right).$$

Approximately optimal weights for the weighted estimator $\widehat{\theta} = \dfrac{1}{W}\sum_{\lambda\in\Lambda} w_\lambda\widehat{\theta}_\lambda$ are given by $w_\lambda = \dfrac{180^2}{\pi^2}\dfrac{\tan^2\delta_{\varepsilon_0}(\lambda)}{1 - \theta^2\tan^2\delta_{\varepsilon_0}(\lambda)}$, where ε_0 is a preliminary estimate of the obliquity.

Note that the calculations are much more complex if we want to estimate the obliquity of the ecliptic underlying a table for the equation of daylight. On the other hand, from a table of a complicated function like the equation of time

$$f_{\varepsilon,\lambda_A,e,c}(\bar{\lambda}) = \tfrac{1}{15}\left(\bar{\lambda} - \alpha_\varepsilon(\bar{\lambda} + q_{e,\lambda_A}(\bar{\lambda})) + c\right),$$

where the right ascension α_ε is defined by $\alpha_\varepsilon(\lambda) = \arctan(\tan\lambda \cdot \cos\varepsilon)$ and the solar equation q_{e,λ_A} by $q_{e,\lambda_A}(\bar{\lambda}) = \arctan\left(\frac{e\sin(\bar{\lambda}-\lambda_A)}{60 + e\cos(\bar{\lambda}-\lambda_A)}\right)$, the obliquity of the ecliptic ε can easily be approximated by means of the weighted estimator, provided that the solar apogee λ_A, the solar eccentricity e and the so-called "epoch constant" c are known. In fact, the calculations are identical to those used to estimate the obliquity of the ecliptic from a right ascension table as explained in the following Section 2.2.2 when λ is replaced by $\bar{\lambda} + q_{e,\lambda_A}(\bar{\lambda})$, y by $y - \bar{\lambda} - c$, and $T(\lambda)$ by $T(\bar{\lambda}) - \bar{\lambda} - c$.

2.2.2 Example: The obliquity of the ecliptic in a right ascension table

The right ascension α_ε is given by $\alpha_\varepsilon(\lambda) = \arctan(\tan\lambda\cdot\cos\varepsilon)$ for $\lambda \in [0, 90\rangle$, where ε is the obliquity of the ecliptic, and follows from the symmetry relations $\alpha_\varepsilon(180-\lambda) = 180-\alpha_\varepsilon(\lambda)$ and $\alpha_\varepsilon(180+\lambda) = 180 + \alpha_\varepsilon(\lambda)$ for $\lambda \in [90, 360]$.[22] Let $T(\lambda)$, $\lambda \in \Lambda$, be a set of tabular values for $\alpha_\varepsilon(\lambda)$ taken from a particular right ascension table. Λ is a finite set of numbers with finite sexagesimal expansion. Note that tabular values for multiples of 90° do not contain any information about the obliquity, since we have $\alpha_\varepsilon(k \cdot 90°) = k \cdot 90°$ for every integer k and every value of ε. Furthermore, in most cases we will use only the tabular values from the first quadrant, since the values, and consequently also the errors, in the other quadrants are probably dependent on those in the first because of the above-mentioned symmetry relations. Therefore we can normally assume that Λ contains only values from the open interval $\langle 0, 90\rangle$.[23] We will furthermore assume that all tabular values were computed using the same (unknown) value of the obliquity, and that the tabular errors $e_\varepsilon(\lambda) \stackrel{\text{def}}{=} T(\lambda) - \alpha_\varepsilon(\lambda)$ are independent and have mean 0 and fixed variance $\sigma^2 > 0$ (the distributions of tabular errors are discussed in more detail in Section 1.2.4).

To simplify the calculations, I will compute the weighted estimator for $\theta \stackrel{\text{def}}{=} \cos\varepsilon$ instead of for ε. Since the historical values of ε range from approximately 23;28 to 24;0, we will have approximately $0.9135 \le \theta \le 0.9173$. Let $g(x,y) = \tan y/\tan x$. Then we have $g(\lambda, \alpha_\varepsilon(\lambda)) = \cos\varepsilon = \theta$ for all $\lambda \in \langle 0, 90\rangle$ and $\theta \in \langle 0, 1]$. Therefore

$$\widehat{\theta}_\lambda \stackrel{\text{def}}{=} g(\lambda, T(\lambda)) = \frac{\tan T(\lambda)}{\tan\lambda} \tag{2.12}$$

is a reasonable estimator for θ for every $\lambda \in \Lambda$. We have

$$\frac{\partial}{\partial y} g(x,y) = \frac{\pi}{180}\frac{1+\tan^2 y}{\tan x} \quad \text{and} \quad \frac{\partial^2}{\partial y^2} g(x,y) = \frac{2\pi^2}{180^2}\frac{\tan y(1+\tan^2 y)}{\tan x}. \tag{2.13}$$

[22] See Section 4.3.8 for more information about the right ascension.

[23] Typical examples of Λ are $\{1, 2, 3, \ldots, 89\}$, $\{6, 12, 18, \ldots, 84\}$ and $\{10, 20, 30, \ldots, 80\}$.

Using formulae (2.3) and (2.4) we find that $\widehat{\theta}_\lambda$ has approximately a distribution with bias

$$\widehat{b}_{\widehat{\theta}_\lambda} = \tfrac{1}{2}\frac{2\pi^2\sigma^2}{180^2}\frac{\tan\alpha_\varepsilon(\lambda)}{\tan\lambda}(1+\tan^2\alpha_\varepsilon(\lambda)) = \frac{\pi^2\sigma^2}{180^2}\theta(1+\theta^2\tan^2\lambda) \tag{2.14}$$

and variance

$$\widehat{S}^2_{\widehat{\theta}_\lambda} = \sigma^2\cdot\left(\frac{\pi}{180}\frac{1+\tan^2\alpha_\varepsilon(\lambda)}{\tan\lambda}\right)^2 = \frac{\pi^2\sigma^2}{180^2}\left(\frac{1+\theta^2\tan^2\lambda}{\tan\lambda}\right)^2. \tag{2.15}$$

From $|\widehat{b}_{\widehat{\theta}_\lambda}|/\widehat{S}_{\widehat{\theta}_\lambda} = \pi\sigma\theta\tan\lambda/180$ we see that the bias of $\widehat{\theta}_\lambda$ becomes more important as λ approaches 90°.[24] For instance, if T is a correct right ascension table to minutes, we have $|\widehat{b}_{\widehat{\theta}_\lambda}|/\widehat{S}_{\widehat{\theta}_\lambda} > 0.1$ if and only if $\lambda > 89°57'21''$.[25] $\widehat{S}_{\widehat{\theta}_\lambda}$ tends to infinity as λ approaches 0° or 90°; this corresponds to the fact that $T(0)$ and $T(90)$ do not contain information about the underlying obliquity value. The minimum value $2\pi\sigma\theta/180$ of $\widehat{S}_{\widehat{\theta}_\lambda}$ is assumed for $\lambda = \arctan(1/\theta) \approx 47.5$. However, as was indicated in Section 2.2.1, even the separate estimator with smallest standard deviation will not allow the unambiguous determination of the obliquity of the ecliptic if the right ascension table under consideration has values to minutes only.

To obtain a more accurate estimate of the obliquity, we will apply the weighted estimator (2.5). Let θ_0 be a reasonable first approximation for θ, e.g. $\theta_0 = \tan T(47)/\tan 47$.[26] As was pointed out in Section 2.2.1, the optimal weights for the weighted estimator are given by $w_\lambda = \dfrac{1}{\operatorname{Var}\widehat{\theta}_\lambda}$, so in this case the weights $w_\lambda = \left(\dfrac{\tan\lambda}{1+\theta_0^2\tan^2\lambda}\right)^2$ are approximately optimal. Thus the weighted estimator $\widehat{\theta}$ is given by

$$\widehat{\theta} = \frac{1}{W}\sum_{\lambda\in\Lambda}\frac{\tan\lambda\cdot\tan T(\lambda)}{(1+\theta_0^2\tan^2\lambda)^2}, \tag{2.16}$$

where $W = \sum_{\lambda\in\Lambda}\left(\dfrac{\tan\lambda}{1+\theta_0^2\tan^2\lambda}\right)^2$. The bias of $\widehat{\theta}$ is approximately equal to

$$\widehat{b}_{\widehat{\theta}} = \frac{\pi^2\sigma^2}{180^2 W}\theta_0\sum_{\lambda\in\Lambda}\frac{\tan^2\lambda}{1+\theta_0^2\tan^2\lambda} \tag{2.17}$$

[24] If we take into account that there is probably a relation between σ^2 and the total number of tabular values n (see Remark 1 of Section 2.2.1) and if we assume that Λ contains equally distributed arguments as below, then we have $\sigma = \sigma_n = \mathcal{O}\left(\frac{1}{n}\right)$, $\tan\lambda \le \tan(90-\frac{1}{n}) = \frac{\cos(\frac{1}{n})}{\sin(\frac{1}{n})} = \mathcal{O}(n)$ for every $\lambda\in\Lambda$, and, consequently, $|\widehat{b}_{\widehat{\theta}_\lambda}|/\widehat{S}_{\widehat{\theta}_\lambda} = \mathcal{O}(1)$ for $n\to\infty$.

[25] As was explained in Section 1.2.4, the rounding errors in this case have approximately a uniform distribution on the interval $\left[-\frac{1}{2}\cdot 60^{-1}, +\frac{1}{2}\cdot 60^{-1}\right]$, hence $\sigma \approx 4.8\cdot 10^{-3}$.

[26] I performed various tests to check that in general θ_0 determined in this way is accurate enough to obtain an approximately optimal estimate for θ. We do have to be careful if $T(47)$ contains a large scribal error (usually these can be recognized by inspecting the error pattern of a preliminary recomputation using the value $\widehat{\theta}$ for the underlying parameter). In that case we may correct or remove the error and calculate a better initial value θ_0 (e.g. from $T(46)$ or $T(48)$). Another possibility is to perform a second weighted estimation of θ with θ_0 taken equal to the result of the first estimation.

and its variance to

$$\widehat{S}_{\widehat{\theta}}^2 = \frac{1}{W^2}\sum_{\lambda\in\Lambda} w_\lambda^2 \widehat{S}_{\widehat{\theta}_\lambda}^2 = \frac{\pi^2\sigma^2}{180^2 W}. \tag{2.18}$$

From equation (2.8) we find that

$$\frac{\left|\widehat{b}_{\widehat{\theta}}\right|}{\widehat{S}_{\widehat{\theta}}} \leq \frac{\pi\sigma\theta_0}{180}\sqrt{\sum_{\lambda\in\Lambda}\tan^2\lambda}. \tag{2.19}$$

Thus the bias of $\widehat{\theta}$ becomes more important if Λ contains a large number of elements in the neighbourhood of 90°.[27]

If we assume that Λ has the specific form $\{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha\}$ for $\alpha = 90/n$, which is the case for most right ascension tables in ancient and mediaeval sources, we can give accurate approximations to W, $\widehat{b}_{\widehat{\theta}}$ and $\widehat{S}_{\widehat{\theta}}$ provided that n is large. Using the integrals given in Appendix 2.A.4 (formulae 2.114 and 2.115) we find[28]

$$W = \sum_{i=1}^{n-1}\left(\frac{\tan\left(\frac{90i}{n}\right)}{1+\theta_0^2\tan^2\left(\frac{90i}{n}\right)}\right)^2 \approx \frac{n}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\tan^2 x\,\mathrm{d}x}{\left(1+\theta_0^2\tan^2 x\right)^2} = \frac{n}{2\theta_0(\theta_0+1)^2} \approx 0.149n, \tag{2.20}$$

$$\begin{aligned}
\widehat{b}_{\widehat{\theta}} &\approx \frac{\pi^2\sigma^2\theta_0}{180^2 W}\sum_{i=1}^{n-1}\frac{\tan^2\left(\frac{90i}{n}\right)}{1+\theta_0^2\tan^2\left(\frac{90i}{n}\right)} \\
&\approx \frac{\pi^2\sigma^2\theta_0}{180^2 W}\cdot\frac{n}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\tan^2 x\,\mathrm{d}x}{1+\theta_0^2\tan^2 x} \\
&\approx \frac{\pi^2\sigma^2\theta_0}{180^2}\cdot\frac{2\theta_0(\theta_0+1)^2}{n}\cdot\frac{n}{\theta_0(\theta_0+1)} \\
&= \frac{2\pi^2\sigma^2\theta_0(\theta_0+1)}{180^2} \approx 1.07\cdot 10^{-3}\sigma^2,
\end{aligned} \tag{2.21}$$

and

$$\widehat{S}_{\widehat{\theta}} = \frac{\pi\sigma}{180\sqrt{W}} \approx \frac{\pi\sigma}{180}(\theta_0+1)\sqrt{\frac{2\theta_0}{n}} \approx \frac{4.53\cdot 10^{-2}\sigma}{\sqrt{n}}. \tag{2.22}$$

Note that from equations (2.21) and (2.22) it follows that[29]

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[27]See, however, formula (2.23) and what is stated below it.

[28]Since in this case we are only interested in the approximate size of the error in $\widehat{\theta}$, we can approximate θ_0 for instance with the cosine of the most common historically attested obliquity value, i.e. $\theta_0 = \cos 23;35 \approx 0.9165$. Within the range of historical values of ε the derived approximations for $\widehat{b}_{\widehat{\theta}}$, $\widehat{S}_{\widehat{\theta}}$ and W vary by less than 1 %.

[29]Using formula 2.113 derived in Appendix 2.A.4, we find that for our choice of Λ the right-hand side of (2.19) equals

$$\frac{\pi\sigma\theta_0}{180}\sqrt{\sum_{i=1}^{n-1}\tan^2\left(\frac{90i}{n}\right)} = \frac{\pi\sigma\theta_0}{180}\sqrt{\frac{(2n-1)(n-1)}{3}} \approx 1.31\cdot 10^{-2}n\sigma,$$

so (2.23) gives us a much sharper upper bound for $|\widehat{b}_{\widehat{\theta}}|/\widehat{S}_{\widehat{\theta}}$ than does (2.19).

$$\frac{\left|\widehat{b}_{\widehat{\theta}}\right|}{\widehat{S}_{\widehat{\theta}}} \approx \frac{\pi\sqrt{2\theta_0 n\sigma}}{180} \approx 2.36 \cdot 10^{-2}\sqrt{n}\sigma. \tag{2.23}$$

We see that, whereas $\widehat{b}_{\widehat{\theta}}$ is a non-zero constant, $\widehat{S}_{\widehat{\theta}}$ approaches 0 for $n \to \infty$. Consequently, $\widehat{b}_{\widehat{\theta}}$ is more important than $\widehat{S}_{\widehat{\theta}}$ when n is large. However, it turns out that if the right ascension table under consideration has correct values rounded to minutes, we have $|\widehat{b}_{\widehat{\theta}}|/\widehat{S}_{\widehat{\theta}} > 0.01$ if and only if $n > 7756$. Since in practice almost all right ascension tables have integer arguments only (hence $n \leq 90$) and have values to at least one sexagesimal fractional digit (hence σ is relatively small), we hardly ever need to take the bias into account. Note that if we assume that the variance of the tabular errors is related to the total number of tabular values n (see Remark 1 of Section 2.2.1), we have $\sigma^2 = \sigma_n^2 = \mathcal{O}\left(\frac{1}{n^2}\right)$ and obtain from equation (2.23):

$$\frac{\left|\widehat{b}_{\widehat{\theta}}\right|}{\widehat{S}_{\widehat{\theta}}} \approx 2.36 \cdot 10^{-2}\sqrt{n}\sigma_n = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad (n \to \infty). \tag{2.24}$$

Hence $\widehat{b}_{\widehat{\theta}}$ is asymptotically negligible for $n \to \infty$.
From

$$\frac{\min\limits_{\lambda} \widehat{S}_{\widehat{\theta}_\lambda}}{\widehat{S}_{\widehat{\theta}}} = \frac{\frac{2\pi\sigma\theta_0}{180}}{\frac{\pi\sigma}{180\sqrt{W}}} \approx 2\theta_0\sqrt{\frac{n}{2\theta_0(\theta_0+1)^2}} \approx \frac{\sqrt{2\theta_0 n}}{\theta_0+1} \approx 0.71\sqrt{n} \tag{2.25}$$

we see that the increase in accuracy of the weighted estimator is proportional to the square root of the number of tabular values. In particular we note that for the common $\Lambda = \{1, 2, 3, \ldots, 89\}$ the weighted estimator has a standard deviation approximately 6.7 times as small as the best separate estimator. Thus the weighted estimator will allow the determination of the value of the obliquity of the ecliptic even if the right ascension values are only given to minutes.

As was indicated in Remark 2 at the end of Section 2.2.1, the approximate bias and variance of the weighted estimator $\widehat{\varepsilon}$ for the obliquity itself can be computed from the bias and variance of $\widehat{\theta}$. Let $h(\theta) = \arccos(\theta) = \varepsilon$. Then we have $h'(\theta) = -\frac{180}{\pi\sqrt{1-\theta^2}}$ and $h''(\theta) = -\frac{180\theta}{\pi(1-\theta^2)^{\frac{3}{2}}}$. Consequently, $\widehat{\varepsilon}$ has approximately bias

$$\widehat{b}_{\widehat{\varepsilon}} = -\frac{180}{\pi\sin\varepsilon}\widehat{b}_{\widehat{\theta}} - \frac{90\cos\varepsilon}{\pi\sin^3\varepsilon}n\widehat{S}^2_{\widehat{\theta}} \approx -143\widehat{b}_{\widehat{\theta}} - 410n\widehat{S}^2_{\widehat{\theta}}, \tag{2.26}$$

where n is the total number of separate estimators, and variance

$$\widehat{S}^2_{\widehat{\varepsilon}} = \left(\frac{180}{\pi\sin\varepsilon}\right)^2 \widehat{S}^2_{\widehat{\theta}} \approx 20.5 \cdot 10^2 \widehat{S}^2_{\widehat{\theta}}. \tag{2.27}$$

In particular, for $\Lambda = \{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha\}$ $(\alpha = 90/n)$ we find an approximate bias $\widehat{b}_{\widehat{\varepsilon}} \approx -0.994\sigma^2$ and an approximate variance $\widehat{S}^2_{\widehat{\varepsilon}} \approx 42.1\sigma^2/n$.

To determine the bias and standard deviation of $\widehat{\theta}$ and $\widehat{\varepsilon}$ for a particular table, it is necessary to approximate the variance σ^2 of the tabular errors using

$$\sigma^2 \approx \frac{1}{n-1} \sum_{\lambda \in \Lambda} \left(T(\lambda) - \alpha_{\widehat{\varepsilon}}(\lambda)\right)^2 , \tag{2.28}$$

where n is the number of elements of Λ and the $\alpha_{\widehat{\varepsilon}}(\lambda)$ are computed values for the right ascension with obliquity equal to the estimate $\widehat{\varepsilon}$. The numbers $T(\lambda) - \alpha_{\widehat{\varepsilon}}(\lambda)$ are the residuals for this particular estimate of ε. In Appendix 2.A.3 it is shown that the right-hand side of equation (2.28) converges in probability to σ^2 for $n \to \infty$.

As was indicated in Section 2.2.1 usually the estimator $\widehat{\theta}$ as given in (2.16) has approximately a normal distribution, provided that the tabular errors $e_\theta(x)$ are independent and have zero means and equal variances, and that the number n of separate estimators $\widehat{\theta}_x$ is large enough. Consequently an approximate 95 % confidence interval for θ is given by

$$\left\langle \widehat{\theta} - \widehat{b}_{\widehat{\theta}} - 1.96 \widehat{S}_{\widehat{\theta}} \; , \; \widehat{\theta} - \widehat{b}_{\widehat{\theta}} + 1.96 \widehat{S}_{\widehat{\theta}} \right\rangle , \tag{2.29}$$

with $\widehat{b}_{\widehat{\theta}}$ as given in (2.17) and $\widehat{S}_{\widehat{\theta}} = \dfrac{\pi\sigma}{180\sqrt{W}}$. An approximate 95 % confidence interval for ε is obtained by taking the arccosine of the bounds of the confidence interval for θ.

A Monte Carlo analysis was carried out to test the validity of (2.29). Confidence intervals were computed from at least 200 computer-generated right ascension tables of different types. These types included tables to minutes, seconds and thirds, whose tabular values were either correct or contained independent uniformly or normally distributed errors with a standard deviation of up to 3 units. For all tables random values of the obliquity within historically plausible limits were used. The results of the Monte Carlo analysis confirmed that (2.29) constitutes an accurate approximate 95 % confidence interval for $\cos\varepsilon$ if the set of arguments Λ equals $\{1, 2, 3, \ldots, 89\}$. 93 % of the estimates lie within the approximate 95 % confidence interval if $\Lambda = \{5, 10, 15, \ldots, 85\}$; 90 % of the estimates if $\Lambda = \{10, 20, 30, \ldots, 80\}$.

An application of the weighted estimator for the obliquity of the ecliptic in a right ascension table can be found in Section 2.6.1.

2.2.3 Example: The solar eccentricity in a solar equation table

The solar equation q_e can be calculated as a function of the mean anomaly $\bar{a}$ according to $q_e(\bar{a}) = \arctan\left(\dfrac{e \sin\bar{a}}{60 + e\cos\bar{a}}\right)$, where e is the solar eccentricity.[30] Let $T(\bar{a})$, $\bar{a} \in \mathcal{A}$, be a set of tabular values for $q_e(\bar{a})$ taken from a particular solar equation table. $\mathcal{A}$ is a finite set of numbers with finite sexagesimal expansion. Since tabular values for multiples

[30] See Section 1.3 for more information about the solar equation.

of 180° do not contain any information about the solar eccentricity and because of the symmetry relation $q_e(\bar{a}) = -q_e(360° - \bar{a})$, we can assume that $\mathcal{A}$ only contains values from the open interval $\langle 0, 180\rangle$. In fact, most tables for the solar equation occurring in ancient and mediaeval sources have double entries for anomalies 0° to 180° and 180° to 360° respectively.[31] We will assume that all tabular values were computed using the same value of the eccentricity, and that the tabular errors $e_e(\bar{a}) \stackrel{\text{def}}{=} T(\bar{a}) - q_e(\bar{a})$ are independent and have mean 0 and fixed variance $\sigma^2 > 0$ (the distributions of tabular errors are discussed in more detail in Section 1.2.4).

As in the case of the right ascension, the calculations can be simplified by computing the weighted estimator for $\theta \stackrel{\text{def}}{=} \frac{60}{e}$ instead of for e. Since we find historically attested values of the solar eccentricity in the range from approximately 1;50 to 2;30, we expect θ to lie between 24 and 33. Let $g(x, y) = \sin x/\tan y - \cos x$. Then we have $g(\bar{a}, q_e(\bar{a})) = \frac{60}{e} = \theta$ for all $\bar{a} \in \langle 0, 180\rangle$ and $\theta > 1$. Therefore

$$\widehat{\theta}_{\bar{a}} \stackrel{\text{def}}{=} g(\bar{a}, T(\bar{a})) = \frac{\sin \bar{a}}{\tan T(\bar{a})} - \cos \bar{a} \tag{2.30}$$

can be used to estimate θ for every $\bar{a} \in \mathcal{A}$. We have

$$\frac{\partial}{\partial y} g(x, y) = -\frac{\pi}{180} \frac{1 + \tan^2 y}{\tan^2 y} \sin x \quad \text{and} \quad \frac{\partial^2}{\partial y^2} g(x, y) = \frac{2\pi^2}{180^2} \frac{1 + \tan^2 y}{\tan^3 y} \sin x. \tag{2.31}$$

Therefore $\widehat{\theta}_{\bar{a}}$ has approximately a distribution with bias

$$\widehat{b}_{\widehat{\theta}_{\bar{a}}} = \tfrac{1}{2} \frac{2\pi^2\sigma^2}{180^2} \frac{1 + \tan^2 q_e(\bar{a})}{\tan^3 q_e(\bar{a})} \sin \bar{a} = \frac{\pi^2\sigma^2}{180^2} \frac{(\theta^2 + 2\theta \cos \bar{a} + 1)(\theta + \cos \bar{a})}{\sin^2 \bar{a}} \tag{2.32}$$

and variance

$$\widehat{S}^2_{\widehat{\theta}_{\bar{a}}} = \sigma^2 \cdot \left(-\frac{\pi}{180} \frac{1 + \tan^2 q(\bar{a})}{\tan^2 q(\bar{a})} \sin \bar{a} \right)^2 = \frac{\pi^2\sigma^2}{180^2} \frac{(\theta^2 + 2\theta \cos \bar{a} + 1)^2}{\sin^2 \bar{a}}. \tag{2.33}$$

Since $\dfrac{\widehat{b}_{\widehat{\theta}_{\bar{a}}}}{\widehat{S}_{\widehat{\theta}_{\bar{a}}}} = \dfrac{\pi\sigma}{180} \dfrac{\theta + \cos \bar{a}}{\sin \bar{a}}$, the bias of $\widehat{\theta}_{\bar{a}}$ becomes more important as $\bar{a}$ approaches 0° or 180°.[32] For instance, if T is a correct solar equation table to minutes for eccentricity 2;4,45, we have $|\widehat{b}_{\widehat{\theta}_{\bar{a}}}|/\widehat{S}_{\widehat{\theta}_{\bar{a}}} > 0.1$ if and only if $\bar{a} < 1°26'12''$ or $\bar{a} > 178°39'34''$. As could be expected, the standard deviation of $\widehat{\theta}_{\bar{a}}$ tends to infinity as $\bar{a}$ approaches 0° or 180°. The minimum possible standard deviation is assumed for $\bar{a} = 90 + \arcsin(2\theta/(1 + \theta^2))$, i.e. close to the maximum of the solar equation at $\bar{a} = 90 + \arcsin(\frac{e}{60})$, and amounts to

[31] The concept of a "table with double entries" is explained in Section 1.1.3. Typical examples of $\mathcal{A}$ are $\{1, 2, 3, \ldots, 179\}$ and $\{6, 12, 18, \ldots, 174\}$.

[32] Assuming that $\sigma = \sigma_n = \mathcal{O}\left(\frac{1}{n}\right)$ (see Remark 1 in Section 2.2.1) and that $\mathcal{A}$ is of the form $\{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha\}$ for $\alpha = 180/n$, we find $(\theta + \cos \bar{a})/\sin \bar{a} \leq 34/\sin\left(\frac{1}{n}\right) = \mathcal{O}(n)$ and hence $|\widehat{b}_{\widehat{\theta}_{\bar{a}}}|/\widehat{S}_{\widehat{\theta}_{\bar{a}}} = \mathcal{O}(1)$ for $n \to \infty$.

$\pi\sigma(\theta^2-1)/180$. If T is a correct solar equation table to minutes based on eccentricity 2;4,35$\frac{1}{2}$ or 2;4,45, the minimum standard deviation of $\widehat{\theta}_{\bar{a}}$ approximately equals $4'11''$, corresponding to a standard deviation of approximately $18''$ for the estimator $\widehat{e}_{\bar{a}}$ for the eccentricity itself. We thus conclude that even the most accurate separate estimator will not be able to distinguish between the two above-mentioned eccentricity values.

Now let θ_0 be a reasonable first approximation for θ, e.g. $\theta_0 = 1/\tan T(90)$. Note that the range of possible values of the eccentricity is rather large, and that we cannot pick an arbitrary historical value for θ_0 in this case.[33] For the weighted estimator $\widehat{\theta}$ defined by $\widehat{\theta} = \frac{1}{W}\sum_{\bar{a}\in\mathcal{A}} w_{\bar{a}}\widehat{\theta}_{\bar{a}}$ we take $w_{\bar{a}} = \dfrac{\sin^2\bar{a}}{(\theta_0^2+2\theta_0\cos\bar{a}+1)^2}$, approximately inversely proportional to $\operatorname{Var}\widehat{\theta}_{\bar{a}}$. We find that $\widehat{\theta}$ has approximately a distribution with bias

$$\widehat{b}_{\widehat{\theta}} = \frac{1}{W}\frac{\pi^2\sigma^2}{180^2}\sum_{\bar{a}\in\mathcal{A}}\frac{\theta_0+\cos\bar{a}}{\theta_0^2+2\theta_0\cos\bar{a}+1} \tag{2.34}$$

and variance

$$\widehat{S}^2_{\widehat{\theta}} = \frac{1}{W^2}\sum_{\bar{a}\in\mathcal{A}} w_{\bar{a}}^2\widehat{S}^2_{\widehat{\theta}_{\bar{a}}} = \frac{\pi^2\sigma^2}{180^2 W}, \tag{2.35}$$

where $W = \sum_{\bar{a}\in\mathcal{A}} w_{\bar{a}}$.

If $\mathcal{A}$ has the common form $\{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha\}$ with n even and $\alpha = 180/n$, we can find accurate approximating expressions for W, $\widehat{b}_{\widehat{\theta}}$ and $\widehat{S}_{\widehat{\theta}}$ provided that n is large. Using the integrals given in Appendix 2.A.4 (formulae 2.116 and 2.117) we find

$$W = \sum_{i=1}^{n-1}\frac{\sin^2\left(\frac{180i}{n}\right)}{\left(\theta_0^2+2\theta_0\cos\left(\frac{180i}{n}\right)+1\right)^2} \approx \frac{n}{\pi}\int_0^\pi \frac{\sin^2 x\,\mathrm{d}x}{(\theta_0^2+2\theta_0\cos x+1)^2} = \frac{n}{2\theta_0^2(\theta_0^2-1)}, \tag{2.36}$$

$$\begin{aligned}\widehat{b}_{\widehat{\theta}} &= \frac{1}{W}\frac{\pi^2\sigma^2}{180^2}\sum_{i=1}^{n-1}\frac{\theta_0+\cos\left(\frac{180i}{n}\right)}{\theta_0^2+2\theta_0\cos\left(\frac{180i}{n}\right)+1}\\ &\approx \frac{\pi^2\sigma^2}{180^2 W}\cdot\frac{n}{\pi}\int_0^\pi\frac{(\theta_0+\cos x)\,\mathrm{d}x}{\theta_0^2+2\theta_0\cos x+1}\\ &\approx \frac{2\pi^2\sigma^2}{180^2}\frac{2\theta_0^2(\theta_0^2-1)}{n}\frac{n}{\theta_0} \approx \frac{2\pi^2\sigma^2\theta_0^3}{180^2}\end{aligned} \tag{2.37}$$

and

$$\widehat{S}_{\widehat{\theta}} = \frac{\pi\sigma}{180W} \approx \frac{\pi\sigma}{180}\sqrt{\frac{2\theta_0^2(\theta_0^2-1)}{n}} \approx \frac{\pi\sigma\theta_0^2}{180}\sqrt{\frac{2}{n}}. \tag{2.38}$$

[33] As in the case of the right ascension, we will have to check whether $T(90)$ contains a scribal error. We may want to repeat the calculation of the weighted estimator with θ_0 taken equal to the obtained estimate $\widehat{\theta}$.

From equations (2.37) and (2.38) it follows that

$$\frac{|\widehat{b}_{\widehat{\theta}}|}{\widehat{S}_{\widehat{\theta}}} \approx \frac{\frac{2\pi^2\sigma^2\theta_0^3}{180}}{\frac{\pi\sigma\theta_0^2}{180}\sqrt{\frac{2}{n}}} = \frac{\pi\sigma\theta_0}{90}\sqrt{\frac{n}{2}}. \tag{2.39}$$

Again $\widehat{b}_{\widehat{\theta}}$ is more important than $\widehat{S}_{\widehat{\theta}}$ when n is large. For a correct solar equation table to minutes for eccentricity 2;4,45, we have $|\widehat{b}_{\widehat{\theta}}|/\widehat{S}_{\widehat{\theta}} > 0.1$ if and only if $n > 851$. Consequently, we may in practice assume that the bias will not overwhelm the standard deviation, but we may not neglect the bias when calculating approximate 95 % confidence intervals. As in the case of the right ascension, we find that $\widehat{b}_{\widehat{\theta}}$ is asymptotically negligible for $n \to \infty$ if we assume that $\sigma = \sigma_n = \mathcal{O}\left(\frac{1}{n}\right)$ (cf. Remark 1 in Section 2.2.1).
From

$$\frac{\min_{\bar{a}} \widehat{S}_{\widehat{\theta}_{\bar{a}}}}{\widehat{S}_{\widehat{\theta}}} = \frac{\frac{\pi\sigma}{180}(\theta_0^2 - 1)}{\frac{\pi\sigma}{180\sqrt{W}}} \approx \frac{\theta_0^2 - 1}{\theta_0^2}\sqrt{\frac{n}{2}} \approx 0.71\sqrt{n} \tag{2.40}$$

we see that the increase in accuracy of the weighted estimator is again proportional to the square root of the number of tabular values involved. Note that the increase is identical to the one found for the right ascension (formula 2.25). For the common $\mathcal{A} = \{1, 2, 3, \ldots, 179\}$ we find that the standard deviation of the weighted estimator is approximately 9.5 times as small as the standard deviation of the most accurate separate estimator. We can conclude that the weighted estimator makes it possible to distinguish even between very close historical values of the eccentricity like 2;4,35$\frac{1}{2}$ and 2;4,45 provided that a sufficient number of tabular values can be used for the estimation.

As was explained in Remark 2 at the end of Section 2.2.1, the approximate bias and variance of the weighted estimator $\widehat{e}$ for the eccentricity itself can be computed from $b_{\widehat{\theta}}$ and $\operatorname{Var}\widehat{\theta}$. We have

$$\widehat{b}_{\widehat{e}} = -\frac{e^2}{60}\widehat{b}_{\widehat{\theta}} + \frac{e^3}{3600}n\widehat{S}^2_{\widehat{\theta}} \qquad \text{and} \qquad \widehat{S}^2_{\widehat{e}} = \frac{e^4}{3600}\widehat{S}^2_{\widehat{\theta}}, \tag{2.41}$$

where n is the total number of separate estimators. We can estimate σ^2 by means of

$$\sigma^2 \approx \frac{1}{n-1}\sum_{\bar{a}\in\mathcal{A}} \left(T(\bar{a}) - q_{\widehat{e}}(\bar{a})\right)^2, \tag{2.42}$$

where n is the number of elements of $\mathcal{A}$ and the $q_{\widehat{e}}(\bar{a})$ are computed values for the solar equation with eccentricity equal to $\widehat{e}$. In Appendix 2.A.3 it is shown that the right-hand side of equation (2.42) converges in probability to σ^2 for $n \to \infty$.

A Monte Carlo analysis similar to the one performed for right ascension tables confirmed that $\left\langle \widehat{\theta} - \widehat{b}_{\widehat{\theta}} - 1.96\,\widehat{S}_{\widehat{\theta}}\ ,\ \widehat{\theta} - \widehat{b}_{\widehat{\theta}} + 1.96\,\widehat{S}_{\widehat{\theta}} \right\rangle$ constitutes an accurate approximate 95 % confidence interval for θ if $\mathcal{A}$ equals $\{1, 2, 3, \ldots, 179\}$. For $\mathcal{A} = \{3, 6, 9, \ldots, 177\}$, $\mathcal{A} = \{5, 10, 15, \ldots, 175\}$ and $\mathcal{A} = \{10, 20, 30, \ldots, 170\}$, 94 % of the estimates lie within the approximate 95 % confidence interval.

An application of the weighted estimator for the eccentricity in a solar equation table can be found in Section 2.6.2.

2.3 The Fourier Estimator for an Unknown Translation Parameter

The Fourier estimator described in this section can be used to estimate the translation parameter of functions satisfying certain general conditions, for example the solar apogee from a table for the solar equation. For the application of the Fourier estimator it is not necessary to know the function that has been tabulated or the values of the other underlying parameters. The estimate is calculated using approximations for Fourier coefficients of the tabulated function, which are computed from the tabular values. Bias and variance of the Fourier estimator will be calculated (Section 2.3.1), and the variance of the tabular errors will be estimated (Section 2.3.3). Three special cases will be discussed in which the tabular errors cannot be assumed to be independent (Section 2.3.2). In two of these cases, the Fourier estimator turns out to have a degenerate distribution. An example of the use of the Fourier estimator can be found in Section 2.6.3.

2.3.1 General Theory

Let g be a 2π-periodic odd function and let f be given by $f(x) = g(x - \lambda_A)$ for every x, where λ_A is an unknown constant.[34] I assume that the Fourier series of f converges, i.e. that, for every $x \in [0, 2\pi]$,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx\right), \tag{2.43}$$

where $a_k = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos kx\,dx$ and $b_k = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin kx\,dx$ for every k. Consequently, the Fourier series of g converges as well, and we can derive the Fourier coefficients of g from those of f. For every $x \in [0, 2\pi]$ we have

$$\begin{aligned} g(x) &= f(x+\lambda_A) \\ &= \tfrac{1}{2}a_0 \\ &\quad + \sum_{k=1}^{\infty} a_k \left(\cos kx \cdot \cos k\lambda_A - \sin kx \cdot \sin k\lambda_A\right) \\ &\quad + \sum_{k=1}^{\infty} b_k \left(\sin kx \cdot \cos k\lambda_A + \cos kx \cdot \sin k\lambda_A\right) \\ &= \tfrac{1}{2}a_0 \\ &\quad + \sum_{k=1}^{\infty} \left(a_k \cos k\lambda_A + b_k \sin k\lambda_A\right) \cos kx \\ &\quad + \sum_{k=1}^{\infty} \left(b_k \cos k\lambda_A - a_k \sin k\lambda_A\right) \sin kx. \end{aligned} \tag{2.44}$$

[34] The notation λ_A is chosen since in practice f will often be a planetary equation and λ_A the longitude of the apogee concerned.

Since g is an odd function, it follows that for every k we have $a_k \cos k\lambda_A + b_k \sin k\lambda_A = 0$, or $\dfrac{\sin k\lambda_A}{\cos k\lambda_A} = -\dfrac{a_k}{b_k}$ (provided that $\cos k\lambda_A \neq 0$). In particular we have $\tan \lambda_A = -\dfrac{a_1}{b_1}$. Now assume that we have a table T of the function f with tabular values $T(i\alpha)$ for $i = 1, 2, 3, \ldots, n$, where n is a multiple of 4 and $\alpha = 2\pi/n$. We will estimate λ_A by first approximating the Fourier coefficients a_1 and b_1 of f by finite sums $\widehat{a}_1$ and $\widehat{b}_1$ computed from the tabular values, and then finding an estimate $\widehat{\lambda_A}$ according to $\widehat{\lambda_A} = -\arctan(\widehat{a}_1/\widehat{b}_1)$ or $\widehat{\lambda_A} = \arctan(\widehat{b}_1/\widehat{a}_1) - 90°$.

First consider the integral $\displaystyle\int_0^{2\pi} h(x)\,dx$ for a 2π-periodic, twice continuously differentiable function h. Let n and α be as above.[35] We have

$$\begin{aligned}
\int_0^{2\pi} h(x)\,dx &= \sum_{i=1}^{n} \int_{(i-\frac{1}{2})\alpha}^{(i+\frac{1}{2})\alpha} h(x)\,dx \\
&= \sum_{i=1}^{n} \int_{(i-\frac{1}{2})\alpha}^{(i+\frac{1}{2})\alpha} \left\{ h(i\alpha) + h'(i\alpha)\cdot(x-i\alpha) + \tfrac{1}{2}h''(\xi_i)\cdot(x-i\alpha)^2 \right\} dx, \\
&= \sum_{i=1}^{n} \left\{ \alpha \cdot h(i\alpha) + 0 + \tfrac{1}{24}\alpha^3 \cdot h''(\xi_i) \right\} \\
&= \frac{2\pi}{n}\sum_{i=1}^{n} h(i\alpha) + \frac{\pi^3}{3n^3} h''(\xi),
\end{aligned} \tag{2.45}$$

where $\xi_i \in \left\langle (i-\frac{1}{2})\alpha, (i+\frac{1}{2})\alpha \right\rangle$ for every $i = 1, 2, 3, \ldots, n$, and $\xi \in [0, 2\pi]$. It follows that the Fourier coefficients $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos x\,dx$ and $b_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin x\,dx$ of the function f can be approximated by

$$\widetilde{a}_1 \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} f(i\alpha)\cos i\alpha \qquad \text{and} \qquad \widetilde{b}_1 \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} f(i\alpha)\sin i\alpha \tag{2.46}$$

respectively. If we write $f_c(x)$ for $f(x)\cos x$ and $f_s(x)$ for $f(x)\sin x$, the errors in the approximations are given by

$$i_a \stackrel{\text{def}}{=} \widetilde{a}_1 - a_1 = \frac{\pi^3}{3n^3} f_c''(\xi) \qquad \text{and} \qquad i_b \stackrel{\text{def}}{=} \widetilde{b}_1 - b_1 = \frac{\pi^3}{3n^3} f_s''(\eta) \tag{2.47}$$

respectively, where ξ and η are numbers in the interval $[0, 2\pi]$. Note that $\widetilde{a}_1$ and $\widetilde{b}_1$ will be better approximations to a_1 and b_1 if the number n of functional values is large and the functions $f(x)\cos x$ and $f(x)\sin x$ are smooth.

Since we do not know the functional values $f(i\alpha)$ themselves, we have to approximate them by the given tabular values $T(i\alpha)$, and we will estimate the Fourier coefficients a_1 and b_1 by

$$\widehat{a}_1 \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\cos i\alpha \qquad \text{and} \qquad \widehat{b}_1 \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\sin i\alpha \tag{2.48}$$

[35] Equation (2.45) actually holds for every positive integer n.

respectively. As usual we will denote the tabular errors $T(i\alpha) - f(i\alpha)$ by $e(i\alpha)$. If e_a denotes the random error of the estimator $\widehat{a}_1$, we have $\widehat{a}_1 = a_1 + e_a + i_a$ with i_a as defined in (2.47), and

$$e_a \stackrel{\text{def}}{=} \widehat{a}_1 - \widetilde{a}_1 = \frac{2}{n}\sum_{i=1}^{n} e(i\alpha)\cos i\alpha. \tag{2.49}$$

Assuming that the tabular errors $e(i\alpha)$, $i = 1, 2, 3, \ldots, n$ are mutually independent and have common mean 0 and variance σ^2, we find that $\widehat{a}_1$ has bias $b_{\widehat{a}_1} = i_a = \frac{\pi^3}{3n^3} f_c''(\xi)$ and variance

$$\begin{aligned} \operatorname{Var}\widehat{a}_1 &= E\left(\frac{2}{n}\sum_{i=1}^{n} e(i\alpha)\cos i\alpha\right)^2 \\ &= \frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \cos i\alpha \cos j\alpha \cdot E(e(i\alpha)e(j\alpha)) \\ &= \frac{4\sigma^2}{n^2}\sum_{i=1}^{n} \cos^2 i\alpha \quad = \quad \frac{4\sigma^2}{n^2}\cdot \tfrac{1}{2}n \quad = \quad \frac{2\sigma^2}{n}. \end{aligned} \tag{2.50}$$

Analogously we write $\widehat{b}_1 = b_1 + e_b + i_b$ with $e_b \stackrel{\text{def}}{=} \widehat{b}_1 - \widetilde{b}_1 = \frac{2}{n}\sum_{i=1}^{n} e(i\alpha)\cos i\alpha$ and find that $\widehat{b}_1$ has bias $b_{\widehat{b}_1} = i_b = \frac{\pi^3}{3n^3} f_s''(\eta)$ and variance $\operatorname{Var}\widehat{b}_1 = 2\sigma^2/n$. Again using the independence of the tabular errors, it follows that

$$\begin{aligned} E(e_a e_b) &= \frac{4}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \cos i\alpha \sin j\alpha \cdot E(e(i\alpha)e(j\alpha)) \\ &= \frac{4\sigma^2}{n^2}\sum_{i=1}^{n} \cos i\alpha \sin i\alpha \quad = \quad 0. \end{aligned} \tag{2.51}$$

It turns out that the constants i_a and i_b usually are very close to zero. For smooth functions like the planetary equations occurring in ancient and mediaeval astronomical handbooks, $|i_a|$ and $|i_b|$ do not exceed 60^{-5} even if n is as small as 12. In order not to make the following calculations of bias and variance of the Fourier estimator unnecessarily complicated, i_a and i_b will therefore be neglected.

Note that there are important cases in which the tabular errors are unlikely to be independent, namely when the apogee to be estimated is equal to $\frac{1}{2}k\alpha$ for an integer k, or when the odd function g defined by $f(x) = g(x - \lambda_A)$ for every x also satisfies the symmetry $g(180 - x) = g(x)$ for every x. In both cases there will probably be pairs of equal or at least correlated tabular values and therefore also of equal or correlated tabular errors. These special cases will be discussed extensively in Section 2.3.2.

Using the Central Limit Theorem, we can easily show that $\widehat{a}_1$ and $\widehat{b}_1$ have approximately normal distributions. For let X_{ni}, $n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots, n$ be defined by $X_{ni} = \sqrt{2}\cos i\alpha \cdot e(i\alpha)/\sqrt{n}\sigma$ and let h_{ni} denote the density of X_{ni}. Then $EX_{ni} = 0$ for every n and i, and $\sum_{i=1}^{n} \operatorname{Var} X_{ni} = \sum_{i=1}^{n} 2\cos^2 i\alpha/n = 1$ for every n. Since, for every $\epsilon > 0$,

$$\max_{i=1,\dots,n} \Pr\left(|X_{ni}| \geq \epsilon\right) \leq \max_{i=1,\dots,n} \frac{\operatorname{Var} X_{ni}}{\epsilon^2} = \max_{i=1,\dots,n} \frac{2\cos^2 i\alpha}{n\epsilon^2} = \frac{2}{n\epsilon^2}, \tag{2.52}$$

it follows that $\max_{i=1,\dots,n} \Pr\left(|X_{ni}| \geq \epsilon\right)$ converges in probability to 0 for $n \to \infty$, and hence that the X_{ni} are uniformly asymptotically negligible. Now the version of the Central Limit Theorem presented in Appendix 2.A.2 states that the distribution of $\sum_{i=1}^{n} X_{ni}$, i.e. of $\sqrt{n}e_a/\sqrt{2}\sigma$, converges to the standard normal distribution if $\lim_{n\to\infty} \sum_{i=1}^{n} \int_{|y|\geq\epsilon} y^2 h_{ni}(y)\,dy = 0$ for every $\epsilon > 0$. This condition holds in most practical cases, in particular whenever the tabular errors $e(i\alpha)$ are uniformly bounded or, more generally, if they have uniformly bounded $2+\delta$ moments for some $\delta > 0$.[36] Similarly, the distribution of $\sqrt{n}e_b/\sqrt{2}\sigma$ converges to the standard normal distribution for $n \to \infty$.

Now let the estimator $\widehat{\theta}$ for $\theta \stackrel{\text{def}}{=} \tan\lambda_A$ be defined by $\widehat{\theta} \stackrel{\text{def}}{=} -\widehat{a}_1/\widehat{b}_1$. We have

$$\begin{aligned}
\widehat{\theta} &= -\frac{a_1+e_a}{b_1+e_b} \\
&= -\frac{a_1+e_a}{b_1}\left(1 - \frac{e_b}{b_1} + \frac{e_b^2}{b_1^2} + \mathcal{O}\left(e_b^3\right)\right) \\
&= \theta - \frac{e_a}{b_1} - \frac{\theta e_b}{b_1} + \frac{e_a e_b}{b_1^2} + \frac{\theta e_b^2}{b_1^2} + \mathcal{O}\left(e_{a,b}^3\right)
\end{aligned} \tag{2.53}$$

for $e_{a,b} \to 0$. It follows that

$$E\widehat{\theta} = \theta + \frac{2\theta\sigma^2}{nb_1^2} + \mathcal{O}(\sigma^3) \qquad \text{and} \qquad \operatorname{Var}\widehat{\theta} = \frac{2(1+\theta^2)\sigma^2}{nb_1^2} + \mathcal{O}(\sigma^3) \tag{2.54}$$

for $\sigma \to 0$. Since $\frac{|b_{\widehat{\theta}}|}{\operatorname{Std}\widehat{\theta}} \approx \frac{\frac{2\theta\sigma^2}{nb_1^2}}{\frac{\sqrt{2}\sqrt{1+\theta^2}\sigma}{\sqrt{n}b_1}} = \frac{\sqrt{2}\theta\sigma}{\sqrt{n}(a_1^2+b_1^2)}$, we find that the estimator $\widehat{\theta}$ is asymptotically unbiased (as usual $b_{\widehat{\theta}}$ denotes the bias of $\widehat{\theta}$, $\operatorname{Std}\widehat{\theta}$ the standard deviation).

Finally, let h denote the function $h(\theta) = \arctan\theta$. An estimator $\widehat{\lambda_A}$ for the translation parameter λ_A can be obtained by putting $\widehat{\lambda_A} \stackrel{\text{def}}{=} h(\widehat{\theta})$.[37] If we write $\widehat{\theta} = \theta + b_{\widehat{\theta}} + e_{\widehat{\theta}}$, then $Ee_{\widehat{\theta}} = 0$ and $\operatorname{Var} e_{\widehat{\theta}} = \operatorname{Var}\widehat{\theta}$, and we have

$$\begin{aligned}
\widehat{\lambda_A} &= h(\theta) + h'(\theta)\cdot(b_{\widehat{\theta}} + e_{\widehat{\theta}}) + \tfrac{1}{2}h''(\theta)\cdot(b_{\widehat{\theta}} + e_{\widehat{\theta}})^2 + \mathcal{O}\left((b_{\widehat{\theta}} + e_{\widehat{\theta}})^3\right) \\
&= \lambda_A + \frac{180}{\pi}\frac{b_{\widehat{\theta}} + e_{\widehat{\theta}}}{1+\theta^2} - \frac{180}{\pi}\frac{\theta}{(1+\theta^2)^2}(b_{\widehat{\theta}} + e_{\widehat{\theta}})^2 + \mathcal{O}\left((b_{\widehat{\theta}} + e_{\widehat{\theta}})^3\right)
\end{aligned} \tag{2.55}$$

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[36] This can be proven in a way analogous to the proof that the linear approximation $\widetilde{\theta}_n$ of the weighted estimator approximately has a normal distribution; see Appendix 2.A.2.

[37] Note that this estimator is identical to $\widehat{\lambda_A} \stackrel{\text{def}}{=} \arctan\left(\widehat{b}_1/\widehat{a}_1\right) - 90$.

for $b_{\widehat{\theta}} + e_{\widehat{\theta}} \to 0$. Consequently,

$$\begin{aligned} E\widehat{\lambda_A} &= \lambda_A + \frac{180}{\pi}\frac{b_{\widehat{\theta}}}{1+\theta^2} - \frac{180}{\pi}\frac{\theta}{(1+\theta^2)^2}(b_{\widehat{\theta}}^2 + \operatorname{Var} e_{\widehat{\theta}}) + \mathcal{O}(\sigma^3) \\ &= \lambda_A + \frac{180}{\pi}\frac{b_1^2}{a_1^2+b_1^2}\cdot\frac{2\theta\sigma^2}{nb_1^2} - \frac{180}{\pi}\frac{\theta b_1^4}{(a_1^2+b_1^2)^2}\cdot\frac{2(a_1^2+b_1^2)\sigma^2}{nb_1^4} + \mathcal{O}(\sigma^3) \\ &= \lambda_A + \mathcal{O}(\sigma^3) \end{aligned} \tag{2.56}$$

and

$$\begin{aligned} \operatorname{Var}\widehat{\lambda_A} &= \frac{180^2}{\pi^2}\frac{\operatorname{Var} e_{\widehat{\theta}}}{(1+\theta^2)^2} + \mathcal{O}(\sigma^3) \\ &= \frac{180^2}{\pi^2}\frac{b_1^4}{(a_1^2+b_1^2)^2}\cdot\frac{2(a_1^2+b_1^2)\sigma^2}{nb_1^4} + \mathcal{O}(\sigma^3) \\ &= \frac{180^2}{\pi^2}\frac{2\sigma^2}{n(a_1^2+b_1^2)} + \mathcal{O}(\sigma^3) \end{aligned} \tag{2.57}$$

for $\sigma \to 0$.

We have seen above that in practice the distributions of $\widehat{a}_1$ and $\widehat{b}_1$ are approximately normal. Whenever $\operatorname{Std}\widehat{b}_1$ is small compared to $\widehat{b}_1$, we expect that $\widehat{\theta} = -\widehat{a}_1/\widehat{b}_1$ has an approximately normal distribution as well. This was confirmed by a Monte Carlo analysis for three different functions with a translation parameter, namely the so-called "method of declinations", the solar equation as a function of the true solar longitude and the solar equation as a function of the mean solar longitude.[38] For each function I calculated tables to minutes and to seconds with correct values or with independent uniformly or normally distributed errors for the sets of arguments $\{1, 2, 3, \ldots, 360\}$, $\{5, 10, 15, \ldots, 360\}$ and $\{30, 60, 90, \ldots, 360\}$. It turned out that, in almost every case, of 1000 tables computed for random values of the underlying parameters within historical limits, between 917 and 957 estimates of the translation parameter were contained in the interval

$$\left\langle \arctan(\widehat{\theta} - b_{\widehat{\theta}} - 1.96\sqrt{\operatorname{Var}\widehat{\theta}})\ ,\ \arctan(\widehat{\theta} - b_{\widehat{\theta}} + 1.96\sqrt{\operatorname{Var}\widehat{\theta}}) \right\rangle, \tag{2.58}$$

with $b_{\widehat{\theta}}$ and $\operatorname{Var}\widehat{\theta}$ as in (2.54). Only for tables with argument set $\{30, 60, 90, \ldots, 360\}$ having values to minutes with small errors, did about 90 % of the estimates lie within the interval.[39]

[38] See Section 1.3 for a description of these functions.

[39] For the "method of declinations" and the solar equation as a function of the true solar longitude, I applied the correction to bias and variance which is described in Section 2.3.2 below.

Remark. Note that λ_A can be estimated from approximated Fourier coefficients $\widehat{a}_k$ and $\widehat{b}_k$ for every k. In fact, for $k > 1$ the calculation of bias and variance of $\widehat{\lambda_A}(k)$ defined by $\widehat{\lambda_A}(k) = -\arctan(\widehat{a}_k/\widehat{b}_k)/k$ is completely analogous to the case $k = 1$. We find that the variance of the estimator $\widehat{\lambda_A}(k)$ is inversely proportional to $k^2(a_k^2 + b_k^2)$. Since in most practical cases a_k and b_k converge rapidly to 0 as k tends to infinity, it follows that the estimates $\widehat{\lambda_A}(k)$ are generally less accurate when k is larger. In Section 2.3.3 it will be shown that the estimators for the Fourier coefficients have zero covariances. Hence we could compute a weighted average of some of the estimates $\widehat{\lambda_A}(k)$ for small k in order to obtain an estimator for λ_A which is better than the estimator $\widehat{\lambda_A}$ discussed above.

2.3.2 Special Cases

As was indicated above, there are special cases of tables for functions with a translation parameter for which the tabular errors $e(i\alpha)$, $i = 1, 2, 3, \ldots, n$ are unlikely to be independent. These cases will now be discussed and the changes in the bias and variance of the estimators $\widehat{a}_1$, $\widehat{b}_1$, $\widehat{\theta}$ and $\widehat{\lambda_A}$ will be calculated.

Case 1. Symmetry of the function g. Assume that the function g as introduced in the preceding section is not only odd, but satisfies the symmetry $g(180 - x) = g(x)$ for every x as well.[40] We then have

$$f(x + 180) = g(x + 180 - \lambda_A) = g(\lambda_A - x) = -g(x - \lambda_A) = -f(x), \tag{2.59}$$

and expect that $T(x + 180) = -T(x)$ and hence $e(x + 180) = -e(x)$ for every x. We obtain

$$e_a = \frac{2}{n} \sum_{i=1}^{\frac{1}{2}n} e(i\alpha) \{\cos i\alpha - \cos(180 + i\alpha)\} = \frac{4}{n} \sum_{i=1}^{\frac{1}{2}n} e(i\alpha) \cdot \cos i\alpha, \tag{2.60}$$

hence $Ee_a = 0$ and $\operatorname{Var} e_a = \frac{16\sigma^2}{n^2} \sum_{i=1}^{\frac{1}{2}n} \cos^2 i\alpha = 4\sigma^2/n$. Similarly, $e_b = \frac{4}{n} \sum_{i=1}^{\frac{1}{2}n} e(i\alpha) \cdot \sin i\alpha$, $Ee_b = 0$, $\operatorname{Var} e_b = 4\sigma^2/n$ and $E(e_a e_b) = 0$. Using equations (2.53) and (2.55), we find

$$E\widehat{\theta} = \theta + \frac{4\theta\sigma^2}{nb_1^2} + \mathcal{O}(\sigma^3) \qquad \text{and} \qquad \operatorname{Var} \widehat{\theta} = \frac{4(1 + \theta^2)\sigma^2}{nb_1^2} + \mathcal{O}(\sigma^3) \tag{2.61}$$

for $\sigma \to 0$. We conclude that the 95 % confidence interval for λ_A is a factor $\sqrt{2}$ larger than we would expect if the tabular errors were mutually independent.

Case 2. Round value of the translation parameter λ_A. Assume that the underlying value of the translation parameter λ_A is of the form $(k + \frac{1}{2})\alpha$ for an integer k, i.e. λ_A falls precisely half-way between two consecutive arguments of the table. Since $f(\lambda_A + x) = g(x) = -g(-x) = -f(\lambda_A - x)$ for every x, we expect for such λ_A that

[40] Examples of functions for which g has this property are the "method of declinations" and the solar equation as a function of the true solar longitude (cf. Section 1.3).

$T(\lambda_A - \frac{1}{2}\alpha + i\alpha) = -T(\lambda_A + \frac{1}{2}\alpha - i\alpha)$ and hence $e(\lambda_A - \frac{1}{2}\alpha + i\alpha) = -e(\lambda_A + \frac{1}{2}\alpha - i\alpha)$ for every i. It follows that

$$\begin{aligned} e_a &= \frac{2}{n}\sum_{i=1}^{\frac{1}{2}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha)\left\{\cos(\lambda_A - \tfrac{1}{2}\alpha + i\alpha) - \cos(\lambda_A + \tfrac{1}{2}\alpha - i\alpha)\right\} \\ &= -\frac{4\sin\lambda_A}{n}\sum_{i=1}^{\frac{1}{2}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha)\cdot\sin(-\tfrac{1}{2}\alpha + i\alpha) \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} e_b &= \frac{2}{n}\sum_{i=1}^{\frac{1}{2}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha)\left\{\sin(\lambda_A - \tfrac{1}{2}\alpha + i\alpha) - \sin(\lambda_A + \tfrac{1}{2}\alpha - i\alpha)\right\} \\ &= \frac{4\cos\lambda_A}{n}\sum_{i=1}^{\frac{1}{2}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha)\cdot\sin(-\tfrac{1}{2}\alpha + i\alpha). \end{aligned} \tag{2.63}$$

Thus $Ee_a = 0$, $\operatorname{Var} e_a = 4\sin^2\lambda_A\sigma^2/n$, $Ee_b = 0$ and $\operatorname{Var} e_b = 4\cos^2\lambda_A\sigma^2/n$. Furthermore, since $e_a = -\tan\lambda_A \cdot e_b$, we find that

$$\widehat{\theta} = -\frac{\widehat{a}_1}{\widehat{b}_1} = -\frac{a_1 + e_a}{b_1 + e_b} = -\frac{-\tan\lambda_A \cdot b_1 - \tan\lambda_A \cdot e_b}{b_1 + e_b} = \tan\lambda_A, \tag{2.64}$$

i.e. $\widehat{\theta}$ has a distribution that degenerates to a point mass at $\tan\lambda_A$. Note that we have $E(e_a e_b) = -\tan\lambda_A \cdot \operatorname{Var} e_b$, and hence that $E(e_a e_b)$ is generally not equal to zero.

If λ_A is of the form $k\alpha$ for an integer k (i.e. λ_A is equal to one of the arguments of the table), the calculations are slightly different. We obtain

$$\begin{aligned} e_a &= \frac{2}{n}\sum_{i=1}^{\frac{1}{2}n-1} e(\lambda_A + i\alpha)\cdot\{\cos(\lambda_A + i\alpha) - \cos(\lambda_A - i\alpha)\} \\ &\quad + \frac{2}{n}\left(e(\lambda_A)\cdot\cos\lambda_A + e(\lambda_A + 180)\cdot\cos(\lambda_A + 180)\right) \\ &= -\frac{4\sin\lambda_A}{n}\sum_{i=1}^{\frac{1}{2}n-1} e(\lambda_A + i\alpha)\cdot\sin i\alpha + \frac{2}{n}\cos\lambda_A(e(\lambda_A) - e(\lambda_A + 180)) \end{aligned} \tag{2.65}$$

and, analogously,

$$e_b = \frac{4\cos\lambda_A}{n}\sum_{i=1}^{\frac{1}{2}n-1} e(\lambda_A + i\alpha)\cdot\sin i\alpha + \frac{2}{n}\sin\lambda_A(e(\lambda_A) - e(\lambda_A + 180)). \tag{2.66}$$

Since $f(\lambda_A) = f(\lambda_A + 180) = 0$, it is reasonable to assume that $T(\lambda_A) = T(\lambda_A + 180) = 0$ and hence $e(\lambda_A) = e(\lambda_A + 180) = 0$. Thus we again have $e_a = -\tan\lambda_A \cdot e_b$ and we find the same biases and variances as above.

CASE 3. SYMMETRY OF THE FUNCTION g COMBINED WITH A ROUND VALUE OF λ_A. Assuming that $g(180 - x) = g(x)$ for every x and that λ_A is of the form $(k + \frac{1}{2})\alpha$ for an integer k, we expect that $e(\lambda_A - \frac{1}{2}\alpha + i\alpha) = e(\lambda_A + 180 + \frac{1}{2}\alpha - i\alpha) = -e(\lambda_A + \frac{1}{2}\alpha - i\alpha)$ for every i. Therefore we obtain

$$e_a = -\frac{8 \sin \lambda_A}{n} \sum_{i=1}^{\frac{1}{4}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha) \cdot \sin(-\tfrac{1}{2}\alpha + i\alpha) \tag{2.67}$$

and

$$e_b = \frac{8 \cos \lambda_A}{n} \sum_{i=1}^{\frac{1}{4}n} e(\lambda_A - \tfrac{1}{2}\alpha + i\alpha) \cdot \sin(-\tfrac{1}{2}\alpha + i\alpha). \tag{2.68}$$

Thus $Ee_a = 0$, $\operatorname{Var} e_a = 8 \sin^2 \lambda_A \sigma^2/n + \mathcal{O}(\frac{1}{n^2})$, $Ee_b = 0$ and $\operatorname{Var} e_b = 8 \cos^2 \lambda_A \sigma^2/n + \mathcal{O}(\frac{1}{n^2})$ for $n \to \infty$. Again we have $e_a = -\tan \lambda_A \cdot e_b$, so $\widehat{\theta} = \tan \lambda_A$ and $\widehat{\lambda_A} = \lambda_A$. If λ_A is of the form $k\alpha$ for an integer k, slightly different calculations lead to the same results.

2.3.3 Variance of the tabular errors

In practice we will mainly apply the Fourier estimator for an unknown translation parameter in situations where we do not know exactly which function has been tabulated or where inexplicable error patterns in the tabular values make it impossible to apply the least squares estimator as described in Section 2.4.[41] This implies that in general we cannot estimate the variance σ^2 of the tabular errors by means of

$$\sigma^2 \approx \frac{1}{n-1} \sum_{i=1}^{n} \left(T(i\alpha) - f(i\alpha)\right)^2. \tag{2.69}$$

I will now indicate two other methods for obtaining an estimate of σ^2, which, contrary to the Fourier estimator itself, make use of the information in estimations for *all* Fourier coefficients a_k for $k = 0, 1, 2, \ldots, \frac{1}{2}n$ and b_k for $k = 1, 2, 3, \ldots, \frac{1}{2}n - 1$.[42] The first method approximates the unknown functional values by means of a finite Fourier expansion. The second method utilizes the fact that for larger k the Fourier coefficients a_k and b_k can

[41] If the tabulated function $f(x) = g(x - \lambda_A)$ (with g odd) is known, it turns out that the least squares estimator and the Fourier estimator give 95 % confidence intervals for the translation parameter of approximately the same width, provided that g does not satisfy the symmetry relation $g(180 - x) = g(x)$ for every x. If g does satisfy this symmetry relation, the 95 % confidence intervals obtained from the least squares estimator are approximately a factor $\sqrt{2}$ smaller than those obtained from the Fourier estimator. We have seen above that in the latter case the standard deviation of the Fourier estimator is a factor $\sqrt{2}$ larger than in the first case.

[42] Note that we cannot estimate more than n Fourier coefficients, since we only have n tabular values at our disposal. The reader may verify that for every $j = 1, 2, 3, \ldots, n$ we have

$$T(j\alpha) = \tfrac{1}{2}\widehat{a}_0 + \sum_{i=1}^{n} \widehat{a}_k \cos j\alpha + \sum_{i=1}^{n} \widehat{b}_k \sin j\alpha + \tfrac{1}{2}\widehat{a}_{\frac{1}{2}n}(-1)^j.$$

be neglected, which implies that the estimates $\widehat{a}_k$ and $\widehat{b}_k$ are practically random variables with zero mean. I have not yet attempted to prove that the resulting estimators for σ^2 are consistent or have other desirable properties. However, in a number of tests it turned out that the relative errors in the estimates of σ^2 obtained by either of the methods were never larger than 10 % except when the number of tabular values was as small as 12 and the number of sexagesimal fractional digits of the table only one. An application of the first method for estimating the variance of the tabular errors can be found in the case study in Section 2.6.3.

At first I will assume that the function g defined by $f(x) = g(x - \lambda_A)$ for every x does not satisfy the symmetry $g(180 - x) = g(x)$. In the same way in which we found approximations for a_1 and b_1 from the tabular values, we can approximate every Fourier coefficient a_k, $k = 0, 1, 2, \ldots, \frac{1}{2}n$ by $\widehat{a}_k = \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\cos ik\alpha$ and every Fourier coefficient b_k, $k = 1, 2, 3, \ldots, \frac{1}{2}n-1$ by $\widehat{b}_k = \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\sin ik\alpha$. It can easily be checked that $\operatorname{Var}\widehat{a}_0 = \operatorname{Var}\widehat{a}_{\frac{1}{2}n} = 4\sigma^2/n$ and that $\operatorname{Var}\widehat{a}_k = \operatorname{Var}\widehat{b}_k = 2\sigma^2/n$ for every $k = 1, 2, 3, ..., \frac{1}{2}n - 1$.[43] Let $e_{\widehat{a}_k} \stackrel{\text{def}}{=} \widehat{a}_k - a_k$ denote the error in $\widehat{a}_k$, $e_{\widehat{b}_k} \stackrel{\text{def}}{=} \widehat{b}_k - b_k$ the error in $\widehat{b}_k$.[44] Using the well-known formulae

$$\sum_{k=0}^{n-1} \cos k\phi = \tfrac{1}{2} + \frac{\sin(n - \frac{1}{2})\phi}{2\sin\frac{1}{2}\phi} \tag{2.70}$$

and

$$\sum_{k=0}^{n-1} \sin k\phi = \frac{\cos\frac{1}{2}\phi - \cos(n - \frac{1}{2})\phi}{2\sin\frac{1}{2}\phi}, \tag{2.71}$$

which hold for every n and for every ϕ which is not a multiple of 2π, it can be shown that $E(e_{\widehat{a}_k}e_{\widehat{a}_l}) = 0$ and $E(e_{\widehat{b}_k}e_{\widehat{b}_l}) = 0$ for every $k \neq l$ and that $E(e_{\widehat{a}_k}e_{\widehat{b}_l}) = 0$ for every k and l.[45] We will assume that n is relatively large.

Method 1. Since a_k and b_k converge to 0 for $k \to \infty$, there will be an integer K such that for all $k \leq K$, $|e_{\widehat{a}_k}|$ and $|e_{\widehat{b}_k}|$ are much smaller than $|\widehat{a}_k|$ and $|\widehat{b}_k|$, and for all $k > K$, $|\widehat{a}_k|$ and $|\widehat{b}_k|$ are of the order of magnitude of $|e_{\widehat{a}_k}|$ and $|e_{\widehat{b}_k}|$ or smaller. We can approximate the functional value $f(x)$ with

$$f_K(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\widehat{a}_0 + \sum_{k=1}^{K} \left(\widehat{a}_k \cos kx + \widehat{b}_k \sin kx\right) \tag{2.72}$$

for every x. The variance σ^2 of the tabular errors can then be estimated by

$$S_1^2 \stackrel{\text{def}}{=} \frac{1}{n-1}\sum_{i=1}^{n} \left(T(i\alpha) - f_K(i\alpha)\right)^2. \tag{2.73}$$

[43] Cf. equation (2.50).

[44] Contrary to the errors e_a and e_b introduced before, $e_{\widehat{a}_k}$ and $e_{\widehat{b}_k}$ do include the error made by approximating the integrals in the definitions of a_k and b_k by finite sums. I will again assume that these errors can be neglected.

[45] Cf. equation (2.51).

Method 2. Let L be an integer such that $|\widehat{a}_k|$ and $|\widehat{b}_k|$ are much smaller than $|e_{\widehat{a}_k}|$ and $|e_{\widehat{b}_k}|$ for all $k \geq L$. Since the mean of $\widehat{a}_k$ and $\widehat{b}_k$ is virtually equal to zero for every $k \geq L$, it follows from the variances given above that

$$\begin{aligned} E\left(\sum_{k=L}^{\frac{1}{2}n} \widehat{a}_k^2 + \sum_{k=L}^{\frac{1}{2}n-1} \widehat{b}_k^2\right) &= \sum_{k=L}^{\frac{1}{2}n} \operatorname{Var} \widehat{a}_k + \sum_{k=L}^{\frac{1}{2}n-1} \operatorname{Var} \widehat{b}_k \\ &= (\tfrac{1}{2}n - L)2\sigma^2/n + 4\sigma^2/n + (\tfrac{1}{2}n - L)2\sigma^2/n \\ &= (2n - 4L + 4)\sigma^2/n. \end{aligned} \tag{2.74}$$

Consequently, the standard deviation σ^2 of the tabular errors can be estimated by

$$S_2^2 \stackrel{\text{def}}{=} \frac{n}{2n - 4L + 4}\left(\sum_{k=L}^{\frac{1}{2}n} \widehat{a}_k^2 + \sum_{k=L}^{\frac{1}{2}n-1} \widehat{b}_k^2\right). \tag{2.75}$$

If the function g defined by $f(x) = g(x - \lambda_A)$ for every x satisfies the symmetry $g(180 - x) = g(x)$, the variance σ^2 of the tabular errors can still be estimated by (2.73) or (2.75). The reader may verify that we have $\widehat{a}_k = \widehat{b}_k = 0$ if k is even, and

$$\widehat{a}_k = \frac{4}{n}\sum_{i=1}^{\frac{1}{2}n} T(i\alpha)\cos ik\alpha \quad \text{and} \quad \widehat{b}_k = \frac{4}{n}\sum_{i=1}^{\frac{1}{2}n} T(i\alpha)\sin ik\alpha \tag{2.76}$$

if k is odd. It follows that $e_{\widehat{a}_k} = e_{\widehat{b}_k} = 0$ if k is even, and that

$$e_{\widehat{a}_k} = \frac{4}{n}\sum_{i=1}^{\frac{1}{2}n} e(i\alpha)\cos ik\alpha \quad \text{and} \quad e_{\widehat{b}_k} = \frac{4}{n}\sum_{i=1}^{\frac{1}{2}n} e(i\alpha)\sin ik\alpha \tag{2.77}$$

if k is odd. Consequently, $\operatorname{Var} \widehat{a}_k = \operatorname{Var} \widehat{b}_k = 4\sigma^2/n$ for every odd k. Finally, it can be checked that the covariances of all $e_{\widehat{a}_k}$ and $e_{\widehat{b}_l}$ for k and l odd are zero.

2.4 Least Squares Estimation

In the preceding two sections two statistical estimators that can be used to approximate the value of a single unknown parameter of an astronomical table have been discussed. For tables for which more than one of the underlying parameter values is unknown, Least Squares estimation turns out to be very useful. In this Section various details of Least Squares estimation will be described. It will be indicated how Least Squares estimates and 95 % confidence intervals for the unknown parameters can be calculated and which assumptions must be satisfied for these to be valid. It will be demonstrated that in certain situations the interpretation of the results of a Least Squares estimation may be problematical.

Let $T(x)$, $x \in \mathcal{X}$ be tabular values for a function f_θ with an unknown parameter vector $\theta \stackrel{\text{def}}{=} (\theta_1, \theta_2, \ldots, \theta_m)^\top$ (here $\top$ denotes the transpose of the vector). Let $e_\theta(x)$ defined by $e_\theta(x) = T(x) - f_\theta(x)$ denote the *residuals* and let the *objective function* $\Phi : \mathbb{R}^m \to \mathbb{R}$ be the sum of the squares of the residuals, i.e.

$$\Phi(\theta) \stackrel{\text{def}}{=} \sum_{x\in\mathcal{X}} e_\theta^2(x) = \sum_{x\in\mathcal{X}} \left(T(x) - f_\theta(x)\right)^2 . \tag{2.78}$$

A parameter vector $\widehat{\theta}$ which minimizes $\Phi(\theta)$ will be called a *least squares estimate* of θ. The *normal equations* for the least squares estimate are given by

$$\left.\frac{\partial \Phi(\theta)}{\partial \theta_i}\right|_{\theta=\widehat{\theta}} = 0 \qquad \text{or} \qquad \sum_{x\in\mathcal{X}} e_{\widehat{\theta}}(x) \frac{\partial f_{\widehat{\theta}}(x)}{\partial \theta_i} = 0 \tag{2.79}$$

for every $i = 1, 2, 3, \ldots, m$. Note that the objective function Φ attaches the same importance to all residuals $e_\theta(x)$. If not all tabular errors are of the same order of magnitude, e.g. in a tangent table, one will want to use so-called *weighted least squares* and will minimize $\sum\limits_{x\in\mathcal{X}} w_x e_\theta^2(x)$ for suitably chosen weights w_x.

If f_θ is a linear function $f_\theta(x) = \theta_1 + \theta_2 x$, the use of least squares leads to a problem of *linear regression*, for which the normal equations can easily be solved.[46] Least Squares estimates for θ_1 and θ_2 are given by

$$\widehat{\theta}_2 = \frac{\sum\limits_{x\in\mathcal{X}} (x - \bar{x}) T(x)}{\sum\limits_{x\in\mathcal{X}} (x - \bar{x})^2} \qquad \text{and} \qquad \widehat{\theta}_1 = \frac{\sum\limits_{x\in\mathcal{X}} T(x)}{n} - \widehat{\theta}_2 \bar{x}, \tag{2.80}$$

where $n \stackrel{\text{def}}{=} \#\mathcal{X}$ denotes the total number of tabular values, and $\bar{x} = \frac{1}{n}\sum\limits_x x$.

If f_θ is non-linear, it is necessary to apply some type of iterative *optimization method* in order to determine a least squares estimate for θ.[47] Starting from an initial parameter vector $\theta^{(0)}$, such methods yield a sequence $\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \ldots$ which is expected to converge to the estimate $\widehat{\theta}$. Most optimization methods determine $\theta^{(k+1)}$ from $\theta^{(k)}$ by first calculating a direction $p^{(k)}$ in which $\Phi(\theta)$ decreases and next finding a positive constant $\alpha^{(k)}$ such that $\Phi(\theta^{(k)} + \alpha^{(k)} p^{(k)})$ is (approximately) as small as possible. $\theta^{(k+1)}$ can then be taken equal to $\theta^{(k)} + \alpha^{(k)} p^{(k)}$.

I will now give brief discussions of two commonly used optimization methods, which are named after Newton and Gauss. For both methods, $q(\theta)$ denotes the gradient of Φ at the point θ, i.e.

$$q(\theta) = \frac{\partial \Phi(\theta)}{\partial \theta} \stackrel{\text{def}}{=} \left(\frac{\partial \Phi(\theta)}{\partial \theta_1}, \ldots, \frac{\partial \Phi(\theta)}{\partial \theta_m} \right)^\top , \tag{2.81}$$

[46] For a discussion of linear regression, see for instance Bickel & Doksum 1977, Chapter 7.

[47] See Draper & Smith 1981, Chapter 10; Bard 1974 or Seber & Wild 1989 for information about nonlinear parameter estimation.

and $H(\theta)$ denotes the Hessian matrix $\dfrac{\partial^2\Phi(\theta)}{\partial\theta^2}$, for which

$$H_{ij}(\theta) = \frac{\partial^2\Phi(\theta)}{\partial\theta_i\partial\theta_j} = 2\sum_{x\in\mathcal{X}} \frac{\partial f_\theta(x)}{\partial\theta_i} \cdot \frac{\partial f_\theta(x)}{\partial\theta_j} - 2\sum_{x\in\mathcal{X}}(T(x) - f_\theta(x))\frac{\partial^2 f_\theta(x)}{\partial\theta_i\partial\theta_j} \tag{2.82}$$

for every $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, m$. The symbols q and H will be used as abbreviations for $q(\theta^{(k)})$ and $H(\theta^{(k)})$ respectively.

Newton's Method. In a neighbourhood of the parameter vector $\theta^{(k)}$ we can approximate $\Phi(\theta)$ with the second order Taylor expansion

$$\Phi^{(k)}(\theta) = \Phi(\theta^{(k)}) + q^\top(\theta - \theta^{(k)}) + \tfrac{1}{2}(\theta - \theta^{(k)})^\top H(\theta - \theta^{(k)}). \tag{2.83}$$

Since $\dfrac{\partial\Phi^{(k)}(\theta)}{\partial\theta} = q+H(\theta-\theta^{(k)})$, a stationary point of $\Phi^{(k)}$ is found for $\theta^{(k)}-H^{-1}q$, provided that H is nonsingular. If H is positive definite, the stationary point is a minimum, and $\theta^{(k)} - H^{-1}q$ can be expected to be closer to the minimum of the function $\Phi(\theta)$ than $\theta^{(k)}$. Thus we define the next step in the iterative procedure to be $\theta^{(k+1)} \stackrel{\text{def}}{=} \theta^{(k)} - H^{-1}q$.

Newton's method converges very rapidly in the neighbourhood of the minimum of $\Phi(\theta)$. On the other hand, away from the minimum the Hessian matrix need not be positive definite, which implies that the method may not converge at all. Another disadvantage is that Newton's method requires the calculation of the second derivatives of f_θ.

Gauss-Newton method. Gauss's modification of Newton's optimization method neglects the quadratic terms of the Hessian matrix and determines the steps of the iteration according to $\theta^{(k+1)} \stackrel{\text{def}}{=} \theta^{(k)} - \alpha^{(k)}\widetilde{H}^{-1}q$ for a certain $\alpha^{(k)} > 0$, where the components of the matrix $\widetilde{H} = \widetilde{H}(\theta^{(k)})$ are given by

$$\widetilde{H}_{ij} = 2\sum_{x\in\mathcal{X}} \left.\frac{\partial f_\theta(x)}{\partial\theta_i}\right|_{\theta=\theta^{(k)}} \cdot \left.\frac{\partial f_\theta(x)}{\partial\theta_j}\right|_{\theta=\theta^{(k)}}. \tag{2.84}$$

Let Q denote the $m \times n$-matrix with components $Q_{ij} = \left.\frac{\partial f_\theta(x_i)}{\partial\theta_j}\right|_{\theta=\theta^{(k)}}$, where the x_i, $i = 1, 2, 3, \ldots, n$ are the elements of $\mathcal{X}$. Since $\widetilde{H} = 2Q^\top Q$, it follows that $\widetilde{H}$ is positive definite. Consequently,

$$\left.\frac{\partial}{\partial\alpha}\Phi(\theta^{(k)} - \alpha\widetilde{H}^{-1}q)\right|_{\alpha=0} = -q^\top\widetilde{H}^{-1}q < 0 \tag{2.85}$$

whenever $q \neq 0$. Thus, contrary to Newton's method, $\Phi(\theta)$ always decreases in the direction of $-\widetilde{H}^{-1}q$. The constant $\alpha^{(k)}$ must be determined in such a way that in fact $\Phi(\theta^{(k)} - \alpha^{(k)}\widetilde{H}^{-1}q) < \Phi(\theta^{(k)})$. Note that close to the minimum of $\Phi(\theta)$ the residuals $e_\theta(x)$ and hence the quadratic terms of the Hessian matrix in formula (2.82) are expected to be small, so the Gauss-Newton optimization method converges as rapidly as Newton's method.

For details concerning the determination of the constant $\alpha^{(k)}$ and convergence and termination of the sequence $\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \ldots$, the reader is referred to textbooks on nonlinear parameter estimation, e.g. Bard 1974 or Seber & Wild 1989. Here I will only note that the value of the found minimum $\Phi(\widehat{\theta})$ should be reasonably small. For example, for a correct table with values to minutes, the variance of the tabular errors is approximately equal to $\frac{1}{12} \cdot 60^{-2}$ (cf. Section 1.2.4) and $\Phi(\widehat{\theta})$ can be expected to be close to $\frac{n}{12} \cdot 60^{-2}$. If $\Phi(\widehat{\theta})$ is much larger than that, we have to consider the possibility that f_θ was not the tabulated function.

Once a least squares estimate $\widehat{\theta}$ has been found, 95 % confidence intervals for the unknown parameters can be determined as follows. Assume that the tabular errors $e_{\widehat{\theta}}(x)$ are independent and have common mean 0 and variance σ^2. It can be shown that the covariance matrix $\operatorname{Cov}\widehat{\theta}$ of the estimated parameter vector can be approximated by

$$\widehat{C}_{\widehat{\theta}} \stackrel{\text{def}}{=} 2\sigma^2 \widetilde{H}(\widehat{\theta})^{-1}, \tag{2.86}$$

where $\widetilde{H}(\widehat{\theta})$ is as defined in equation (2.84) with $\theta^{(k)}$ replaced by $\widehat{\theta}$.[48] The variance of the tabular errors can be estimated by

$$\sigma^2 \approx \frac{1}{n-m} \sum_{x \in \mathcal{X}} e^2_{\widehat{\theta}}(x) = \frac{1}{n-m} \Phi(\widehat{\theta}). \tag{2.87}$$

If we can assume that the estimates $\widehat{\theta}_i$ for the separate parameters are unbiased and have a normal distribution, we find that approximate 95 % confidence intervals for the parameters θ_i are given by

$$\left\langle \widehat{\theta}_i - 1.96 \operatorname{Std}\widehat{\theta}_i \,,\, \widehat{\theta}_i + 1.96 \operatorname{Std}\widehat{\theta}_i \right\rangle, \tag{2.88}$$

where $\operatorname{Std}\widehat{\theta}_i = \sqrt{(\widehat{C}_{\widehat{\theta}})_{ii}}$. In order to spot possible dependences between the parameters, we will also want to compute and plot a *confidence region* for the parameter vector θ. Again assuming that the estimates $\widehat{\theta}_i$ are unbiased and have normal distributions, we find that $(\theta - \widehat{\theta})^\top \widehat{C}^{-1}_{\widehat{\theta}} (\theta - \widehat{\theta})$ has a χ^2 distribution with m degrees of freedom.[49] It follows that if c_m is the upper 95 % point of that distribution, then the ellipsoid

$$(\theta - \widehat{\theta})^\top \widehat{C}^{-1}_{\widehat{\theta}} (\theta - \widehat{\theta}) \leq c_m \tag{2.89}$$

is an approximate 95 % confidence region for θ. I performed a large number of Monte Carlo tests on tables of different types and found that the confidence intervals and confidence regions obtained in this way actually contained the parameters to be estimated in approximately 19 out of 20 cases.[50]

[48] See Bard 1974, pp. 176–179.
[49] See Bard 1974, pp. 187–189.
[50] I performed the Monte Carlo tests in particular for tables with multiple underlying parameters,

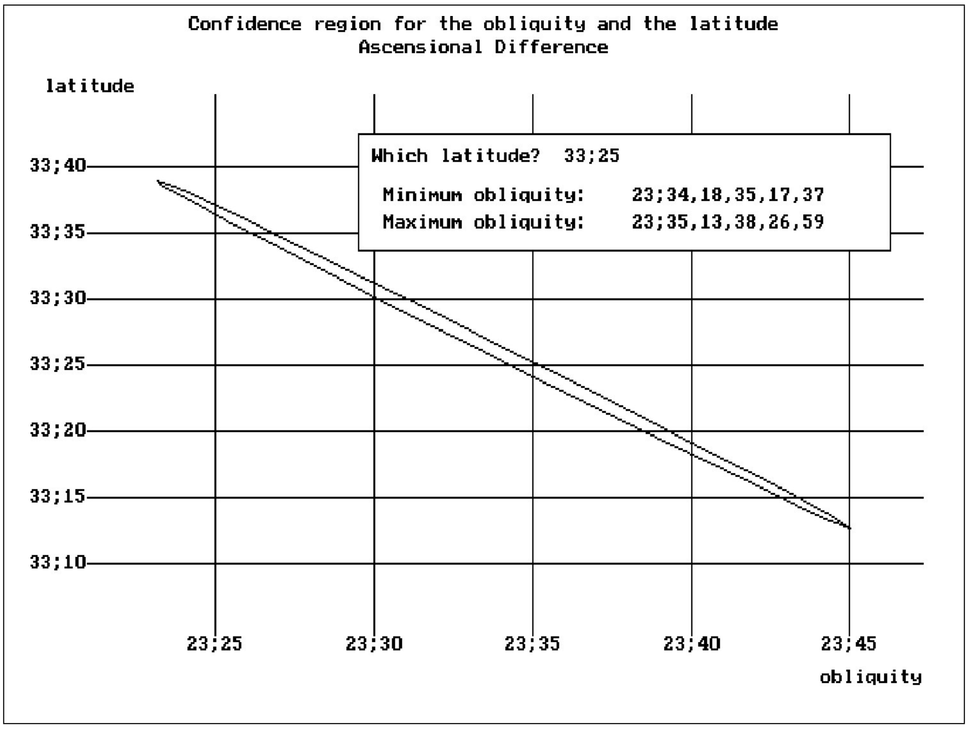

Figure 2.1: 95 % confidence region for the parameters of an table for the equation of daylight

Note that if we estimate a single unknown parameter by means of least squares, the estimated variance is equal to

$$\operatorname{Var}\widehat{\theta} = 2\sigma^2\widetilde{H}^{-1} = \frac{\sigma^2}{\sum\limits_{x\in\mathcal{X}}\left(\frac{\partial f_\theta(x)}{\partial\theta}\right)^2}. \tag{2.90}$$

Since $\theta \to f_\theta(x)$ is the inverse function of $y \to g(x,y)$, where g is defined as in Section 2.2.1, we have $\dfrac{\partial f_\theta(x)}{\partial\theta} = \dfrac{1}{\frac{\partial g(x,y)}{\partial y}}$, and it follows that the variance of the least squares estimate as found above is equal to the variance of the weighted estimator as given in formula (2.6).

In my computer program Table-Analysis described in Section 1.4.1 I incorporated the Gauss-Newton method in order to estimate the unknown parameter values from astronomical tables by means of least squares. It turned out that generally the Gauss-Newton

namely the equation of daylight (see Section 4.3.10), the oblique ascension (see Section 4.3.13), the equation of time as a function of the true solar longitude, and the equation of time as a function of the mean solar longitude (see Section 3.1.1). Most of the tests were carried out on tables with an argument increment of 5 or 6 degrees and with values to minutes or seconds, which were either correct or contained small independent tabular errors from a Gaussian or uniform distribution. In each case 200 estimations were performed. In general, the confidence regions as determined above seemed to contain the underlying parameter values in somewhat less than 19 out of 20 cases. Only if the number of tabular values was smaller than approximately 20 and if the number of sexagesimal fractional digits was so small that consecutive rounding errors became correlated, did the marginal confidence intervals as given above contain the underlying parameter values in only 18 out of 20 cases.

method converges rapidly to the desired minimum. If the initial parameter vector $\theta^{(0)}$ is not extremely far away from the minimum of $\Phi(\theta)$, usually only three or four iterations are required to approximate $\widehat{\theta}$ with sufficient accuracy. For most tables that I have analysed, the covariances of the estimated parameter values turn out to be small and inferences about the parameters can be made from the marginal confidence intervals. However, for functions like the equation of daylight $\Delta_{\varepsilon,\phi}(\lambda) = \arcsin(\tan\delta_\varepsilon(\lambda)\cdot\tan\phi)$, where the solar declination δ_ε is defined by $\delta_\varepsilon(\lambda) = \arcsin(\sin\lambda\cdot\sin\varepsilon)$, the estimates of parameters ε and ϕ turn out to be strongly correlated. As can be seen from Figure 2.1, the 95 % confidence region determined according to the method presented above is then very narrow. Consequently, at opposite ends of the confidence region we can find pairs of parameter values that are quite far removed but lead to equally good recomputations of the equation of daylight table.[51] If we can decide on a particular historical value for one of the two parameters (like the geographical latitude $\phi = 33;25$ of Baghdad in the case of the figure), the other parameter value is expected to lie within a cross-section of the confidence region which is much smaller than the marginal confidence interval (see the "cross-section window" in the figure; the marginal 95 % confidence interval for ε in this case is $\langle\, 23;25,21\,,\ 23;42,51\,\rangle$).

Applications of the estimation of multiple unknown parameters by means of the method of least squares can in particular be found in Sections 2.6.3, 3.2, 3.3, 3.4 and 4.3.14.1.

2.5 Least Number of Errors Criterion

At various places in this thesis[52] I make explicit or implicit use of what I call the "Least Number of Errors Criterion": a single unknown parameter θ of a given table is approximated by a value $\widehat{\theta}$ for which the number of errors in the table, i.e. the number of differences between the table and a recomputation, is smallest. In this section a short, informal discussion of the Least Number of Errors Criterion (to be abbreviated as *LNEC*) is presented. I will indicate how a statistical justification of the use of the *LNEC* can be given, but I have not developed a complete statistical theory.

Let $T(x)$, $x \in \mathcal{X}$ be tabular values for a function f_θ which depends on a parameter θ. I will assume that the tabular values were rounded in the modern way and will denote this rounding by r_u, where u is the unit of the tabular values.[53] Assume that there exists

[51] Section 4.3.14.1 shows an example of an equation of daylight table for which the least squares estimates point to the use of $\varepsilon = 23;51$ and $\phi = 32;0$. However, a recomputation on the basis of $\varepsilon = 23;35$, which occurs throughout the zīj under consideration, and $\phi = 32;20$, which is mentioned in the heading of the table, turns out to yield a recomputation which is almost as good.

[52] See in particular the examples in Sections 2.6.1 and 2.6.2.

[53] As was explained in Section 1.1.3, the "unit of the tabular values" is the greatest common divisor of all tabular values. As an example, a table with values to (sexagesimal) seconds has unit 60^{-2}. If, as a result of the method of calculation, the number of seconds of all tabular values is a multiple of 4, the unit is $4\cdot 60^{-2}$. Note that if the table T is correct, we have $r_u(f_\theta(x)) = T(x)$ for every $x \in \mathcal{X}$. The theory presented in this section can be modified for the case where truncation or rounding up was applied instead of modern rounding by adding $\frac{1}{2}\delta$ to or subtracting $\frac{1}{2}\delta$ from all tabular values.

a function g: $\mathbb{R}^2 \to \mathbb{R}$ such that $g(x, f_\theta(x)) = \theta$ for all x and θ.[54] For every $x \in \mathcal{X}$ and $\delta > 0$, let the interval $\Theta_{x,\delta}$ be given by $[g(x, T(x) - \frac{1}{2}\delta), g(x, T(x) + \frac{1}{2}\delta)\rangle$ if $y \to g(x, y)$ is increasing, or $\langle g(x, T(x) + \frac{1}{2}\delta), g(x, T(x) - \frac{1}{2}\delta)]$ if $y \to g(x, y)$ is decreasing.[55] Then $\Theta_{x,\delta}$ consists of all values $\widehat{\theta}_x$ of the parameter θ for which the functional value $f_{\widehat{\theta}_x}(x)$ differs by less than $\frac{1}{2}\delta$ from the tabular value $T(x)$. In particular, if δ is equal to the unit u of the tabular values, $\Theta_{x,\delta}$ consists of all parameter values $\widehat{\theta}_x$ for which $r_u(f_{\widehat{\theta}_x}(x)) = T(x)$.

Now for given values of θ and $\delta > 0$, let $\mathcal{X}_{\theta,\delta} = \{x \in \mathcal{X} | f_\theta(x) \notin \Theta_{x,\delta}\}$ be the set of arguments x for which the tabular value $T(x)$ is "in error". Here $T(x)$ will be called "in error" whenever, for the given values of θ and δ, $f_\theta(x)$ differs by more than $\frac{1}{2}\delta$ from $T(x)$. In particular, if δ is equal to the unit u of the table, $\mathcal{X}_{\theta,\delta}$ consists of all arguments for which the tabular values are different from recomputed values for the given value of θ.

Finally, let the set Θ_δ consist of those parameter value(s) $\widehat{\theta}$ for which $\#\mathcal{X}_{\widehat{\theta},\delta}$ is as small as possible. Then every $\widehat{\theta} \in \Theta_\delta$ minimizes the number of tabular values that are "in error" and will be called an estimate for the unknown parameter θ according to the Least Number of Errors Criterion.

Remark 1. The name of the *LNEC* derives from the case where δ is taken equal to the unit u of the tabular values. As was indicated above, the set $\mathcal{X}_{\widehat{\theta},\delta}$ then consists of all arguments x for which the tabular value $T(x)$ is not equal to the recomputed value $f_{\widehat{\theta}}(x)$ based on parameter value $\widehat{\theta}$, i.e. for which $r_u(f_{\widehat{\theta}}(x)) \neq T(x)$. If all tabular values are correct, we have $\mathcal{X}_{\widehat{\theta},\delta} = \varnothing$ and $\Theta_\delta = \bigcap_{x \in \mathcal{X}} \Theta_{x,\delta}$, and the parameter value θ used for the computation of the table is contained in Θ_δ.

Remark 2. The *LNEC* is insensitive to large scribal errors: if an otherwise correct table contains a number of outliers, then the set Θ_δ will be the same as when the outliers were left out completely. The reason for this is that in general the underlying value of the parameter θ will be far removed from the interval $\Theta_{x,\delta}$ for any outlying tabular value $T(x)$, which implies that the corresponding arguments x will be contained in $\mathcal{X}_{\theta,\delta}$ for every θ close to the underlying parameter value.

The *LNEC* as presented above is in the first place a numerical criterion for determining reasonable approximations for an unknown parameter value. By assuming certain distributions of the tabular errors $e_\theta(x) \stackrel{\text{def}}{=} T(x) - f_\theta(x)$ we can give a statistical justification of the *LNEC*. As was shown in Section 1.2.4, in many practical cases rounding errors can be assumed to be independent and have a uniform distribution. Thus, if u is the unit of a correctly computed and rounded table, the density h of the tabular errors $e_\theta(x)$ can be assumed to be

$$h(e) = \begin{cases} 2u^{-1} & e \in \left\langle -\frac{1}{2}u, +\frac{1}{2}u \right] \\ 0 & \text{otherwise.} \end{cases} \tag{2.91}$$

[54]This is the same function g used for the weighted estimator. Cf. Section 2.2.1 and footnote 17.

[55]Usually δ will be taken equal to the unit of the table under consideration. I expect that if a table contains many errors, it may be useful to choose δ larger than the unit, but have not tried this.

function[56]	underlying parameter	argument increment 1°	argument increment 5°
solar declination	obliquity	2″	9″
right ascension	obliquity	10″	54″
length of longest day	obliquity	8″	35″
solar equation $\bar{q}(\bar{a})$	eccentricity	1″	6″

Table 2.2: Average lengths of the interval Θ_δ

One can now show that if δ equals u, the set Θ_δ consists of all maximum likelihood estimates for the parameter θ.[57]

In practice every table contains at least a couple of errors. Therefore it is more realistic to assume a density of the tabular errors of the following type:

$$h(e) = \begin{cases} \eta & e \in \left\langle -\frac{1}{2}u, +\frac{1}{2}u \right] \\ \xi & e \in \left[a, -\frac{1}{2}u \right] \cup \left\langle +\frac{1}{2}u, b \right] \\ 0 & \text{otherwise,} \end{cases} \tag{2.92}$$

where again u is the unit of the table, $a \leq -\frac{1}{2}u$, $b \geq +\frac{1}{2}u$, $u\eta + (b - a - u)\xi = 1$, $u\eta$ is close to 1, and $0 < \xi << \eta$. Thus most of the errors are expected to fall within the interval $\langle -\frac{1}{2}u, +\frac{1}{2}u]$, a small number of errors are expected to fall outside this interval but within $[a, b]$. Now if δ equals u, one can show that again the set Θ_δ consists of all maximum likelihood estimates for the parameter θ.

Table 2.2 displays the average length of the interval Θ_δ for various types of tables and for two different argument increments. In each case the average was taken over 200 tables with correct values to minutes, based on random values for the underlying parameter within historically plausible limits. δ was taken equal to the unit of the tables, namely 60^{-1}. It can be noted that in all cases the length of the interval Θ_δ is significantly smaller than the length of the 95 % confidence interval obtained from the weighted estimator.[58]

We conclude that the Least Number of Errors Criterion can be a very useful tool for determining unknown parameter values, especially for tables with very few errors. More research is necessary to exploit the possibilities of the *LNEC* in full. In particular, it would be interesting to determine confidence intervals for the unknown parameter, and to investigate the behaviour of the *LNEC* when applied to tables with larger numbers of errors.

[56] See Section 1.3 for references to descriptions of these functions.

[57] For an explanation of the maximum likelihood estimator, see Bickel & Doksum 1977, pp. 99–107. See also van Dalen 1989, Appendix A4, pp. 130–132.

[58] For argument increment 1° the average length of Θ_δ is approximately fives times smaller than the length of the 95 % confidence intervals, for argument increment 5°, two or three times smaller.

2.6 Examples

2.6.1 The Obliquity of the Ecliptic in the Sanjufīnī Zīj[59]

In the Bibliothèque Nationale in Paris there is a unique copy of a zīj completed in 1366 by ʿAṭā ibn Aḥmad al-Sanjufīnī.[60] The author originated from the region of Samarkand in Central Asia, but his work is dedicated to the viceroy of Tibet, a direct descendant of Genghis Khan.[61] Consequently, the zīj is a very interesting mixture of Islamic, Mongolian and Tibetan influences.

At least three different values for the obliquity of the ecliptic are incorporated in the Sanjufīnī Zīj. Firstly, Kennedy and Hogendijk showed that the tables for parallax on folio 42^{v} and for determining the visibility of the lunar crescent on folio 38^{r} were computed using the Ptolemaic value 23;51.[62] Secondly, the common Islamic value 23;35 underlies the table for the solar altitude on folio 40^{r}.[63] Thirdly, the solar declination table on folio 37^{v} is based on $\varepsilon = 23;32,30$, a value which has so far been found only in the zīj of Ibn Isḥāq al-Tūnisī,[64], and which is very close to the 23;32,29 associated with the Alphonsine tables.[65] Since there is as yet no historical reason for assuming a relationship between Ibn Isḥāq and the Alphonsine tables on the one hand and al-Sanjufīnī on the other, the value 23;32,30 may in the present context be considered as typical for al-Sanjufīnī. The same holds for the geographical latitude $38°10'$, which is explicitly mentioned in the headings of five tables in the Sanjufīnī Zīj. Kennedy and Hogendijk showed that this value is indeed involved in the table for determining the visibility of the lunar crescent.[66] It can also easily be seen that the latitude value $38°10'$ underlies the table for the solar altitude, which does not mention the value explicitly.[67] Finally, it corresponds to a locality in Tibet which is mentioned in the manuscript.[68]

In this section the right ascension table and one of the two oblique ascension tables in the Sanjufīnī Zīj will be investigated. The weighted estimator as presented in Section 2.2.2 will be used to determine the values of the obliquity of the ecliptic underlying these

[59]Section 2.6.1 was published in van Dalen 1989, pp. 98–106. A small number of unimportant changes have been made.

[60]Paris Bibliothèque Nationale Ms. Arabe 6040, 57 folios, 14th century. The complete name of the author is given as Abū Muḥammad ʿAṭā ibn Aḥmad ibn Muḥammad ibn Khwāja Ghāzī al-Samarqandī al-Sanjufīnī (folio 2^{v}; in Wüstenfeld 1866–1870, vol. 3, pp. 162 a village Sanjafīn close to Samarkand is mentioned. I do not know why Kennedy nevertheless used the transcription Sanjufīnī.). At the bottom of folio 26^{v} it is indicated that the manuscript is an autograph. Furthermore the date of completion is given as 13 Rabīʿ II 768 A.H. (17 December 1366). The date 764 A.H. given by Blochet (see Blochet 1925, p. 169) occurs in connection with the epoch of the planetary tables in the zīj. The following articles deal with aspects of the Sanjufīnī Zīj: Kennedy 1987/88, Kennedy & Hogendijk 1988 and Franke 1988.

[61]See Kennedy & Hogendijk 1988, p. 233.

[62]See Kennedy & Hogendijk 1988, pp. 237 and 241.

[63]This follows from the tabular values 28;15,0 for 0° Capricorn and 51;50,0 for 0° Aries.

[64]Hyderabad Andra Pradesh State Library Ms. 298, table 54.

[65]See Rico y Sinobas 1864, vol. 3, pp. 296–297.

[66]See Kennedy & Hogendijk 1988, p. 241.

[67]This follows from the tabular value for 0° Aries given in footnote 63.

[68]See Kennedy 1987/88, pp. 61–62.

λ'	$T(\lambda')$	λ'	$T(\lambda')$	λ'	$T(\lambda')$
1	1; 5	31	33;15	61	63; 4
2	2;11	32	34;17	62	64; 1
3	3;16	33	35;19	63	64;58
4	4;22	34	36;21	64	65;55
5	5;27	35	37;23	65	66;52
6	6;32	36	38;24	66	67;48
7	7;38	37	39;26	67	68;45
8	8;43	38	40;27	68	69;41
9	9;48	39	41;28	69	70;37
10	10;53	40	42;29	70	71;33
11	11;58	41	43;29	71	72;29
12	13; 3	42	44;30	72	73;25
13	14; 8	43	45;30	73	74;21
14	15;13	44	46;30	74	75;17
15	16;18	45	47;30	75	76;12
16	17;23	46	48;30	76	77; 8
17	18;28	47	49;29	77	78; 3
18	19;31	48	50;28	78	78;59
19	20;35	49	51;27	79	79;54
20	21;39	50	52;26	80	80;49
21	22;43	51	53;25	81	81;44
22	23;47	52	54;24	82	82;40
23	24;51	53	55;22	83	83;35
24	25;55	54	56;20	84	84;30
25	26;58	55	57;18	85	85;25
26	28; 1	56	58;16	86	86;20
27	29; 4	57	59;14	87	87;15
28	30; 7	58	60;12	88	88;10
29	31;10	59	61;10	89	89; 5
30	32;13	60	62; 7	90	90; 0

Table 2.3: Transcription of the normed right ascension table on folio 39^{r} of the Sanjufīnī Zīj.

tables. The "least number of errors criterion" (see Section 2.5) will be applied to check the results. On the basis of the numerical results, historical conclusions will be drawn and the origin of the tables will be discussed.

Right ascension table. The Sanjufīnī Zīj has a table for the normed right ascension on folio 39^{r}.[69] For the analysis of this table we will use only the first 89 tabular entries, which will be denoted by $T(\lambda')$, $\lambda' = 1, 2, 3, \ldots, 89$.[70] By comparing these entries with the symmetrical values in the other quadrants, we can correct a single scribal error (see the Apparatus in Section 2.6.4). A transcription of the first quadrant of the corrected table can be found in Table 2.3. From the tabular value $T(43)$ we obtain the most accurate approximation for the obliquity calculated from a single tabular value. A recomputation of al-Sanjufīnī's table for the normed right ascension yields the same tabular value for argument 43 that we find in the manuscript whenever ε is contained in the interval ⟨23;33,34 , 23;38,8⟩. Within this interval, 23;35 is the only frequently used attested value. Nevertheless, because of the possibility of a scribal or computational error in $T(43)$ and since other obliquity values are found in the Sanjufīnī Zīj as well, it seems advisable to compute a more accurate approximation.

The weighted estimator for the obliquity of the ecliptic in a right ascension table as given in formula (2.16) in Section 2.2.2 yields $\widehat{\varepsilon} = 23;34,41$. By means of formula (2.29) we find that an approximate 95 % confidence interval for ε is given by ⟨23;34,21 , 23;35,14⟩, provided that the conditions regarding the tabular errors are satisfied. This does not seem to be the case, since the errors with respect to recomputations for obliquity values within the confidence interval occur in small groups in which the errors have the same sign. The statistical tests indicated in Section 1.2.4 do indeed show that the errors cannot be considered to be independent.

To determine a valid 95 % confidence interval, we will apply the weighted estimator to a set of values for which the tabular errors do satisfy the condition of independence. In the case of a table computed by means of interpolation, a natural choice for such a set are the independently computed tabular values.[71] In the present table, however, no traces of interpolation can be found: there is no set of equidistant tabular values giving significantly smaller tabular errors than the intermediate values, and the finite differences of the tabular values do not display regular patterns.

Another possible means of achieving independence of the tabular errors is to omit coherent parts of the table in which the errors are highly dependent, e.g. parts where the tabulated function is almost linear or where many errors with the same sign occur. An example of such a situation is found in the solar equation table in the Shāmil Zīj (see Section 2.6.2). In the present case, however, the above-mentioned small groups of errors with respect to recomputation are distributed all over the table, and omitting a coherent part of the table does not remove the dependence.

[69] More information about the normed right ascension can be found in Section 1.3.

[70] Note that $\lambda' = \lambda + 90°$. As was indicated in Section 2.2.2, the tabular errors for arguments 91 to 360 are probably dependent on those for arguments 1 to 90, and can therefore not be used for the weighted estimator as developed in that section.

[71] For instance, in the case of linear interpolation, neighbouring tabular values tend to have errors which have the same sign and therefore are dependent.

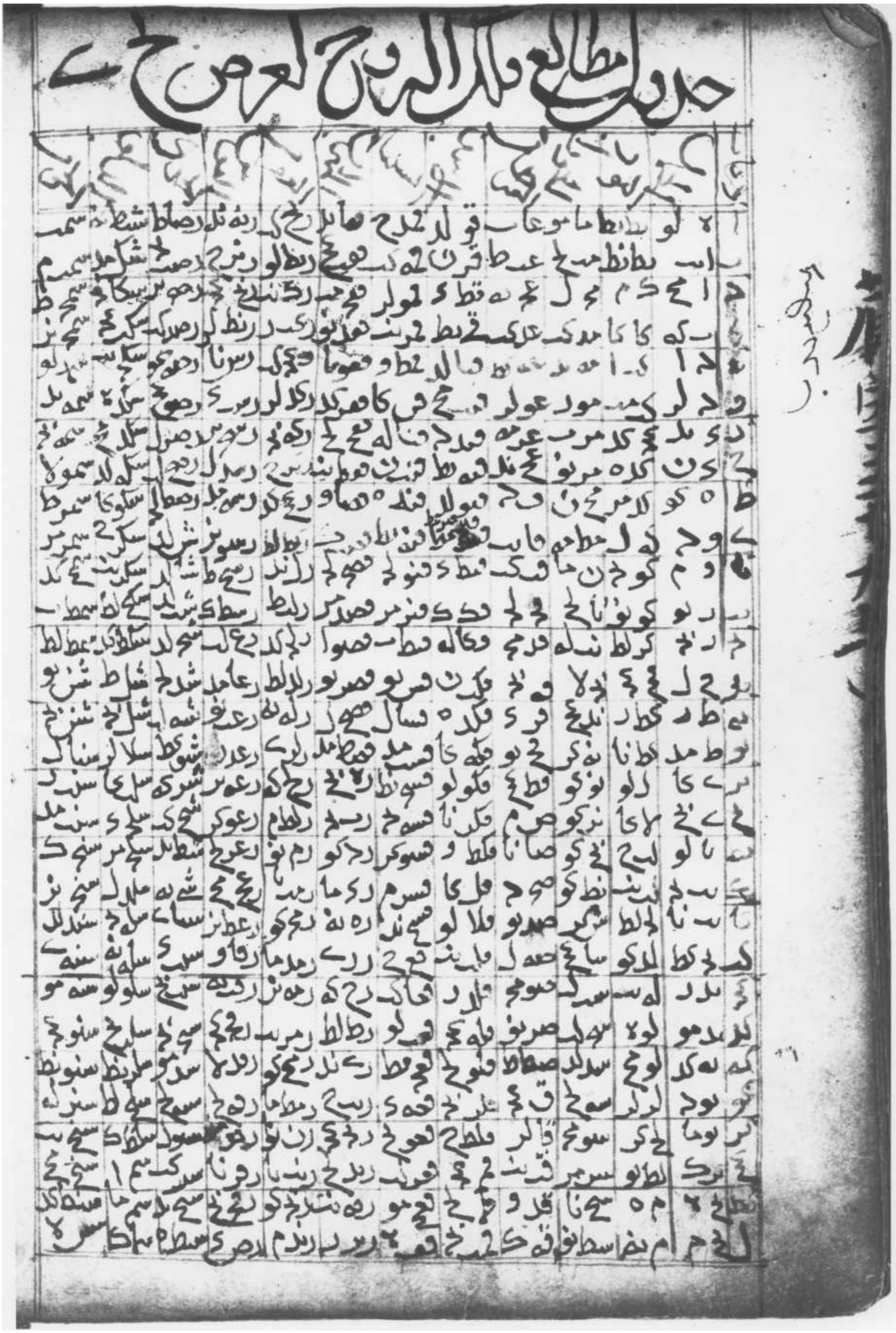

Plate 2.1: The oblique ascension table on folio 38^{v} of the Sanjufīnī Zīj

A third possiblity is to compute the weighted estimator for subtables $T(\lambda_0+k\cdot\Delta\lambda)$, $k = 0, 1, 2, \ldots$, where $\Delta\lambda$ is fixed and $\lambda_0 \in \{1, 2, \ldots, \Delta\lambda\}$.[72] Since a number of entries in the original table are thus left out, the tabular errors in such subtables are expected to show a less obvious mutual relationship, i.e. to be less dependent. In fact, it appears that for the right ascension table in the Sanjufīnī Zīj the tabular errors for both even ($\Delta\lambda = 2, \lambda_0 = 2$) and odd ($\Delta\lambda = 2, \lambda_0 = 1$) arguments pass the test for independence. Monte Carlo tests for tables with these sets of arguments show that in both cases (2.29) constitutes an accurate approximate 95 % confidence interval. Now for even arguments we have $\widehat{\varepsilon}$ = 23;34,50 and a 95 % confidence interval is given by ⟨23;34,12 , 23;35,27⟩; for odd arguments we have $\widehat{\varepsilon}$ = 23;34,45 and a 95 % confidence interval is given by ⟨23;34,7 , 23;35,23⟩. As no other frequently used attested value of the obliquity is contained in either confidence interval, we conclude that for this table ε = 23;35. This result is confirmed by means of the "least number of errors criterion" (see Section 2.5). The minimum possible number of errors with respect to recomputation is eight and is reached for $\varepsilon \in$ ⟨23;34,39,24 , 23;34,55,51⟩. For $\varepsilon \in$ ⟨23;34,38,25 , 23;34,39,24⟩ and $\varepsilon \in$ ⟨23;34,55,51 , 23;35,2,55⟩, the number of errors is nine.

The nine errors with respect to the recomputation for ε = 23;35 are given in the Apparatus in Section 2.6.4. Eight are between 30 and 40 seconds in absolute value and could easily have resulted from small inaccuracies in the intermediate steps of the computation. In this way, only the error for $\lambda' = 17$ would remain unexplained. Because of the observed dependence of the tabular errors, I also tried several linear interpolation schemes. It turned out that none of them gave a better fit than exact computation.

Oblique ascension table. The oblique ascension ρ is given by $\rho(\lambda) = \alpha(\lambda) - \Delta(\lambda)$, where $\Delta(\lambda) = \arcsin(\tan\delta(\lambda)\cdot\tan\phi)$ is the equation of daylight, dependent both on the obliquity ε (through the declination $\delta(\lambda) = \arcsin(\sin\lambda\cdot\sin\varepsilon)$) and on the geographical latitude ϕ.[73] By means of the identity $\rho(\lambda) - \rho(180-\lambda) = 2\cdot\alpha(\lambda) - 180$, it is possible to "extract" the right ascension used for the computation of a given oblique ascension table. In the same manner the equation of daylight can be extracted using $\rho(\lambda) + \rho(180-\lambda) = 180 - 2\Delta(\lambda)$.

The Sanjufīnī Zīj has two oblique ascension tables, of which I shall study the one on folio 38$^{\text{v}}$ (see Plate 2.1). From its heading it appears that this table was computed for ϕ = 38;10. The right ascension values that can be extracted from it are displayed in Table 2.4. I will denote the tabular values of the oblique ascension table by $T_\rho(\lambda)$, $\lambda = 1, 2, 3, \ldots, 360$, and the extracted right ascension values by $T_\alpha(\lambda)$, $\lambda = 1, 2, 3, \ldots, 90$.

It can be noted that the values for both the right ascension and the equation of daylight used for the computation of the present oblique ascension table were given to seconds (or even more accurately). For if at least one of the two were given to minutes, $T_\rho(\lambda) + T_\rho(180-\lambda)$ would have an even number of minutes for all λ, provided that the standard rounding procedure were used.[74] For the present table, however, $T_\rho(\lambda) + T_\rho(180-\lambda)$ is odd for about half of the values of λ.

[72] Thus, for example, the estimator could be computed from every second or from every third tabular value.

[73] For more information about the oblique ascension, see Section 4.3.13 of this thesis; Kennedy 1956a, p. 140; and Pedersen 1974, pp. 99–101 and 110–115.

[74] In case of systematic truncation or rounding up, $T_\rho(\lambda) + T_\rho(180-\lambda)$ would have an odd number of

λ	$T_\alpha(\lambda)$	λ	$T_\alpha(\lambda)$	λ	$T_\alpha(\lambda)$
1	0;55, 0	31	28;50,30	61	58;50, 0
2	1;50, 0	32	29;48, 0	62	59;53, 0
3	2;45, 0	33	30;46, 0	63	60;56,30
4	3;40,30	34	31;44, 0	64	61;59,30
5	4;36, 0	35	32;41,30	65	63; 2,30
6	5;30,30	36	33;39,30	66	64; 5,30
7	6;26, 0	37	34;38,30	67	65; 9,30
8	7;21, 0	38	35;36,30	68	66;13, 0
9	8;16, 0	39	36;35,30	69	67;17, 0
10	9;11,30	40	37;34,30	70	68;21, 0
11	10; 6,30	41	38;33,30	71	69;25, 0
12	11; 1,30	42	39;32,30	72	70;29, 0
13	11;57, 0	43	40;31,30	73	71;33,30
14	12;53, 0	44	41;31, 0	74	72;37,30
15	13;48,30	45	42;31, 0	75	73;42, 0
16	14;44, 0	46	43;30,30	76	74;47, 0
17	15;39,30	47	44;30,30	77	75;51,30
18	16;35,30	48	45;30,30	78	76;56,30
19	17;31,30	49	46;32, 0	79	78; 2, 0
20	18;27, 0	50	47;31,30	80	79; 7, 0
21	19;23, 0	51	48;32,30	81	80;12, 0
22	20;19,30	52	49;33,30	82	81;17, 0
23	21;16, 0	53	50;34,30	83	82;22,30
24	22;12,30	54	51;36, 0	84	83;27,30
25	23; 9, 0	55	52;37, 0	85	84;32,30
26	24; 5,30	56	53;39, 0	86	85;38, 0
27	25; 2, 0	57	54;41,30	87	86;44, 0
28	25;59, 0	58	55;43, 0	88	87;49, 0
29	26;56, 0	59	56;45,30	89	88;54,30
30	27;53,30	60	57;47,30	90	90; 0, 0

Table 2.4: Right ascension extracted from the oblique ascension table on folio 38^{v}

If for a particular λ neither $T_\rho(\lambda)$ nor $T_\rho(180-\lambda)$ is in error, the extracted value $T_\alpha(\lambda)$ differs from the correct right ascension value $\alpha(\lambda)$ by not more than 0;0,30. If the tabular errors of $T_\rho(\lambda)$ have a uniform distribution, the tabular errors of $T_\alpha(\lambda)$ will have a triangular distribution.[75] It is not sufficient to approximate the obliquity of the ecliptic underlying the extracted right ascension from the single value that yields the most accurate approximation, namely $T_\alpha(47)$: all values of ε in the interval ⟨23;31,16 , 23;35,51⟩ yield a right ascension value for $\lambda = 47$ which differs from $T_\alpha(47)$ by less than 0;0,30. In particular, the two values 23;32,30 and 23;35, which are attested in the Sanjufīnī Zīj itself, are both contained in this interval.

The weighted estimator for the obliquity of the ecliptic in a right ascension table yields $\widehat{\varepsilon}$ = 23;32,48, and a 95 % confidence interval is given by ⟨23;32,23 , 23;33,13⟩, provided that the conditions for its validity are satisfied. However, as in the preceding case, the tabular errors turn out to be insufficiently independent.

Therefore I computed the weighted estimator for subtables $T(\lambda_0+k\cdot\Delta\lambda)$, $k = 0, 1, 2, \ldots$ of the original table. For $\Delta\lambda = 3$ the tabular errors of the subtables do pass the test for independence. Monte Carlo tests show that (2.29) constitutes an accurate 95 % confidence interval for such subtables. We have:

subtable	*estimate* $\widehat{\varepsilon}$	*confidence interval*
$\lambda_0 = 0$	23;32,49	⟨ 23;32,10 , 23;33,27 ⟩
$\lambda_0 = 1$	23;32,26	⟨ 23;31,39 , 23;33,13 ⟩
$\lambda_0 = 2$	23;33, 9	⟨ 23;32,26 , 23;33,52 ⟩

Three different attested values for the obliquity are contained in all three confidence intervals: 23;32,29 (Alphonsine tables), 23;32,30 (Ibn Isḥāq and al-Sanjufīnī) and 23;33 (Mumtaḥan Zīj).[76] Since only the second of these values is found elsewhere in the Sanjufīnī Zīj, we conclude that very probably 23;32,30 is the value underlying the extracted right ascension table. This conclusion is confirmed by the result of the "least number of errors criterion": the minimum possible number of differences larger than 0;0,30 with a recomputation is five, obtained for $\varepsilon \in$ ⟨23;32,26,44 , 23;32,35,14⟩. The five differences larger than 0;0,30 for ε = 23;32,30 are given in the Apparatus in Section 2.6.4.

Conclusions. *The right ascension table on folio 39*$^{\text{r}}$ *of the Sanjufīnī Zīj was computed using* ε = 23;35. *The right ascension used for the computation of the oblique ascension table on folio 38*$^{\text{v}}$ *was computed using* ε = 23;32,30.

A preliminary investigation showed that the equation of daylight extracted from the oblique ascension table on folio 38$^{\text{v}}$ was computed using ϕ = 38;10 (in agreement with the information in the heading of the table) and ε = 23;35. Thus this oblique ascension

minutes for all λ if either the right ascension or the equation of daylight were given to minutes. If both were given to minutes, $T_\rho(\lambda) + T_\rho(180-\lambda)$ would be even for all λ.

[75] This implies that the tabular errors of the extracted right ascension have a smaller standard deviation than those of the oblique ascension. For instance, a uniform distribution on the interval $[-\frac{1}{2}, +\frac{1}{2}]$ has standard deviation $\frac{1}{6}\sqrt{3} \approx 0.29$, a triangular distribution on the same interval has standard deviation $\frac{1}{12}\sqrt{6} \approx 0.20$.

[76] See footnotes 64 and 65, and Kennedy 1956a, p. 145.

table involves *two different* values of the obliquity of the ecliptic.
The second oblique ascension table in the Sanjufīnī Zīj, which is found on folio 28ᵛ, is based on $\varepsilon = 23;32,30$ (consistently) and $\phi = 32;0$.

Since the two oblique ascension tables in the Sanjufīnī Zīj involve the typical obliquity value 23;32,30, it can be concluded that both are either by al-Sanjufīnī himself, or were taken by him from the source from which he took his solar declination table. It seems plausible that al-Sanjufīnī or his source had a right ascension table based on $\varepsilon = 23;32,30$. It is unclear why al-Sanjufīnī did not include this table in his zīj.

The equation of daylight used for the computation of the oblique ascension table on folio 38ᵛ may have come from a different source, since it involves a different obliquity value. On the other hand, the latitude $38°10'$ on which it is based occurs four other times in the Sanjufīnī Zīj. As was indicated before, this latitude corresponds to a locality in Tibet mentioned in the manuscript.[77] It is unclear why al-Sanjufīnī also included two tables for latitude 32° in his zīj.[78]

2.6.2 The Solar Eccentricity in the Shāmil Zīj[79]

The Shāmil Zīj is an anonymous Arabic zīj extant in at least eight copies.[80] In spite of the fact that it was obviously frequently used, it has not been investigated systematically. Like the contemporary Baghdādī,[81] Ashrafī[82] and Muṣṭalaḥ[83] Zījes, it was not an original work, but was based largely upon earlier sources now partially or completely lost. I

[77] See footnote 68.

[78] In addition to the oblique ascension table on folio 28ᵛ, the parallax table on folio 28ʳ, which is explicitly attributed to al-Sanjufīnī, involves this value. The most important localities to which a latitude of 32° is attributed are Jerusalem and Isfahan; see Kennedy & Kennedy 1987, pp. 678–679.

[79] Section 2.6.2 was published in van Dalen 1989, pp. 106–113. A small number of unimportant changes have been made.

[80] Paris Bibliothèque Nationale: Ms. Arabe 2528 (73 folios, 15th century), Ms. Arabe 2529 (76 folios, 16th century), Ms. Arabe 2540 (folios 29ᵛ–97ᵛ, 15th century); Florence Laurenziana: Or. 95 [289] (116 folios, 15th century), Or. 106–1 (folios 1ᵛ–71ʳ, early 14th century, Mosul); Cairo Dār al-Kutūb Ṭalʿat mīqāt 138 (115 pages, c. 1500), Taimūr riyāḍ 296/1 (pages 1–160, 18th century); and London British Library II,2 Ms. 395/3 (Add. 7492 Rich.; 67 folios, c. 1500). A commentary on the Shāmil Zīj is found in Paris Bibliothèque Nationale Ms. Arabe 2530. See de Slane 1883–95, pp. 451–452 and 454; Suter 1902, pp. 166–167; Kennedy 1956a, p. 129, no. 29; *GAS*, vol. 5, pp. 324–325; and King 1986b, p. 52, no. B100.

[81] The Baghdādī Zīj was compiled shortly before the year 1285 by Jamāl al-Din al-Baghdādī. It is extant in the unique manuscript Paris Bibliothèque Nationale Ms. 2486 (225 folios, 1285 A.D.) and contains material of earlier astronomers such as Ḥabash and Abu'l-Wafāʾ. More information about the Baghdādī Zīj and an analysis of most of the trigonometric and spherical astronomical tables in the zīj can be found in Chapter 4 of this thesis.

[82] The Ashrafī Zīj was written in Persian by Muḥammad Sanjar al-Kamālī (Shiraz, south-western Persia, c. 1300). It gives the mean motion parameters and planetary equations from a large number of earlier zījes and is extant in Paris Bibliothèque Nationale Ms. Suppl. Persan 1488 (288 folios, 1303 A.D.). See Kennedy 1956a, p. 124, no. 4; and Kennedy 1977, p. 183.

[83] The anonymous Muṣṭalaḥ Zīj was used in Egypt between the 13th and 15th centuries. It is extant in Paris Bibliothèque Nationale Ms. 2513 (94 folios, 13th century) and Ms. 2520 (175 folios, 14th century). See de Slane 1883–95, pp. 446 and 448–449; Kennedy 1956a, pp. 131–132, no. 47; and King 1983, pp. 535–536.

inspected the tables in four of the Shāmil manuscripts mentioned in footnote 80 and also those in the Dublin manuscript of the zīj of Athīr al-Dīn al-Abharī (fl. Mardin, c. 1240).[84] It turned out that the Shāmil Zīj and the zīj of al-Abharī are essentially the same. As Kennedy pointed out, both zījes involve the mean motion parameters found by Abu'l-Wafāʾ al-Būzjānī (see below).[85] The epoch of the mean motion tables in the Shāmil Zīj is the year 600 of the Yazdigird era (1231 A.D.). From this it may be concluded that the zīj was compiled shortly after this date, which would fit very well with the dates of al-Abharī's scholarly activity. The mean motion tables indicate that they were computed for longitude 84°, and the values of the geographical latitude occurring in the latitude-dependent tables are 37°, 38° and 39°. The conclusion that the Shāmil Zīj was used in north-western Persia is confirmed by the fact that the localities in the geographical table are concentrated in this region.[86] The Chester Beatty manuscript of al-Abharī's zīj has an oblique ascension table stated to be specifically for Mardin (folio 16^v). Its title mentions a latitude of 37°25′ and a longitude of 75°0′.[87] However, the tables for the mean planetary positions are the same as those in the Shāmil Zīj, and their headings likewise mention the longitude 84°.[88] Therefore it seems plausible that al-Abharī simply copied them from the Shāmil Zīj.

The above-mentioned Abu'l-Wafāʾ lived in Baghdad in the 10th century and compiled an astronomical handbook called al-Majisṭī (Almagest). A substantial part of this work is extant in a Paris manuscript.[89] This unique copy contains important trigonometric material, which was analysed by Carra de Vaux,[90] but it does not contain any tables. A zīj called Wāḍiḥ Zīj is also attributed to Abu'l-Wafāʾ, but is not extant.[91] The relation between the tables in the Wāḍiḥ Zīj and those belonging to al-Majisṭī is unknown. More information about Abu'l-Wafāʾ's tables must be obtained from related zījes such as the Shāmil.

An extensive analysis of the manuscripts of the Shāmil Zīj is beyond the scope of this case study. I will restrict myself to applying the weighted estimator introduced

[84] Dublin Chester Beatty Ms. 4076 (61 folios, c. 1400). Information about al-Abharī can be found in Suter 1900, pp. 145–146, no. 364, and in the article by C. Brockelmann in EI_2.

[85] Kennedy 1956a, pp. 129 and 133. Information about Abu'l-Wafāʾ can be found in Section 4.1.2 of this thesis; in Suter 1900, pp. 71–72, no. 167; and in the article by A.P. Yushkevich in the Dictionary of Scientific Biography (*DSB*).

[86] The geographical table is given in Kennedy & Kennedy 1987, pp. 471–473.

[87] These coordinates are not found in the list given in Kennedy & Kennedy 1987, pp. 216–217. However, al-Battānī's values $\phi = 37°15'$ and $\lambda = 75°0'$ are very close. It can be shown that the oblique ascension table in the zīj of al-Abharī was based on $\phi = 37°0'$, and not on $\phi = 37°15'$ or $\phi = 37°25'$.

[88] Note that the difference in longitude cannot be explained from different meridians of reference. Firstly, both the oblique ascension table in the zīj of al-Abharī and the geographical table in the Shāmil Zīj state that the longitudes are measured from the Fortunate Isles. Secondly, the longitude 75° could in principle refer to north-western Persia (when measured from the "western shore of the encompassing sea"), but in al-Abharī's zīj it occurs only in explicit connection with Mardin. See Kennedy & Kennedy 1987, p. xi for a description of the meridians of reference.

[89] Paris Bibliothèque Nationale Ms. 2494 (107 folios, 12th century). See de Slane 1883–95, p. 442.

[90] Carra de Vaux 1892.

[91] See Kennedy 1956a, p. 134, no. 73; and Suter 1892, pp. 39–40. Suter translates "al-zīj al-wāḍiḥ" as "das Buch der genauen (klaren, zweifellosen) Tafeln".

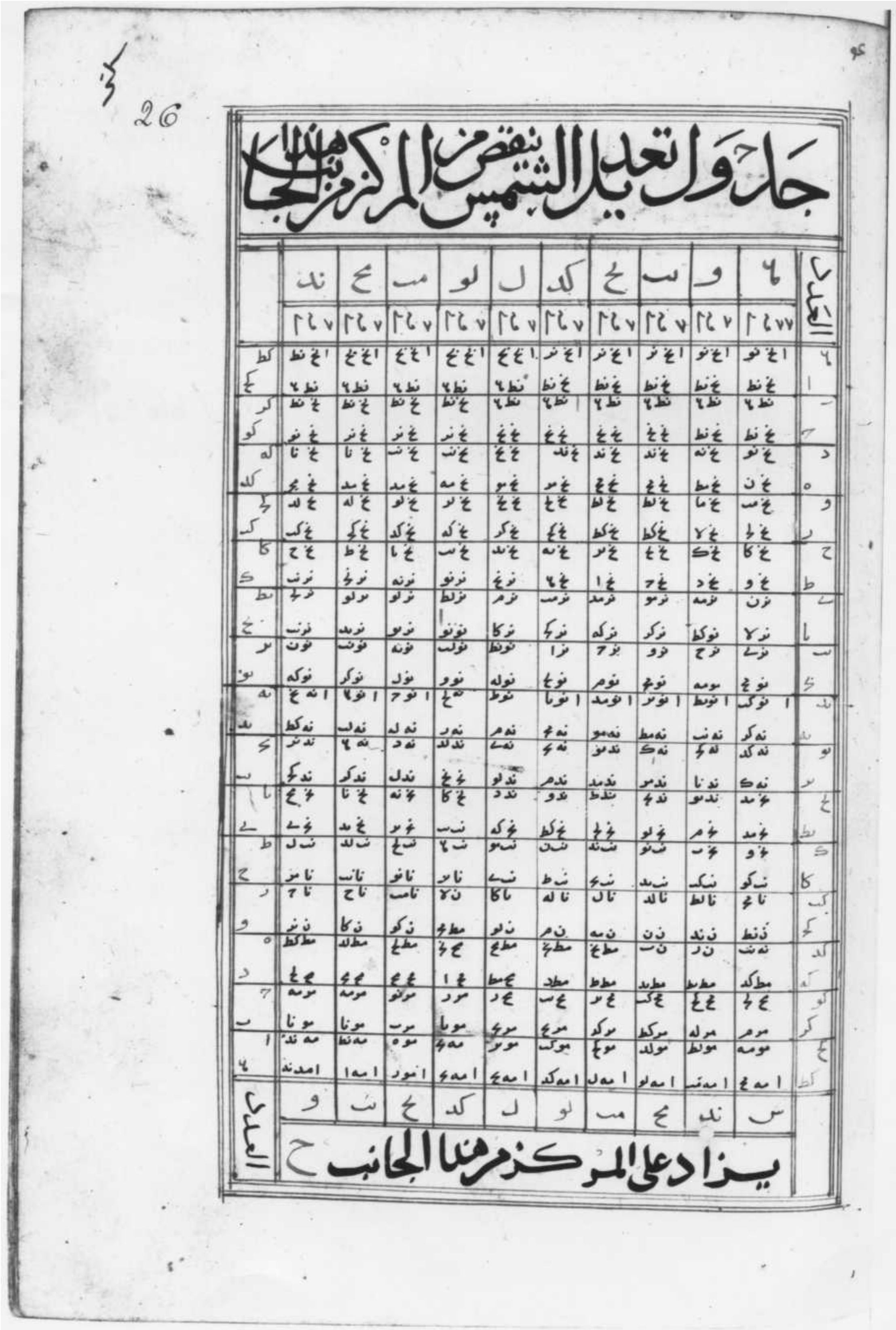

Plate 2.2: Part of the solar equation table in the Shāmil Zīj (Paris BN 2528, folio 26^{r})

in Section 2.2.3 to a single table. It will be shown how, by means of this estimator, both the underlying parameter value and the accuracy used in the computation can be determined. Once the table has been "classified" in this way, we will be able to comment on its authorship.

Plate 2.2 shows one page of the solar equation table found on folios 24^{v}–27^{r} of Paris Bibliothèque Nationale Ms. Arabe 2528. The table displays the solar equation to seconds for every six minutes of the mean solar anomaly. However, especially in the neighbourhood of the maximum (which occurs at 92°) it is clear that the tabular values in between integer numbers of the argument were computed by means of linear interpolation. Note that the interpolated values were rounded in the modern way, thus sexagesimal digits 6, 12, 18 and 24 were rounded down, whereas digits 30, 36, 42, 48 and 54 were rounded up. Because of the regularity of the interpolation pattern it is easy to correct the scribal errors in the tabular values for integer arguments. The errors are given in the Apparatus in Section 2.6.4, the corrected values in Table 2.5.

We will use the tabular entries for integer values of the solar anomaly to find an accurate approximation for the solar eccentricity that was used for the computation of the table. At first sight it is clear that the table was intended for the attested maximum solar equation of $1°59'$,[92] which occurs in the table for a mean solar anomaly of 92°. However, all values of the eccentricity between 2;4,34,59 and 2;4,36,1 lead to a recomputed value $T(92) = 1;59,0$. Therefore it is not possible to conclude from the single tabular value $T(92)$ whether 2;4,35 or 2;4,36 or some value in between was used.[93]

If all tabular values for integer arguments are used for the computation of the weighted estimator, we find $\widehat{e} = 2;4,35,29$ according to formula (2.5) in Section 2.2.1. Provided that the conditions regarding the tabular errors are satisfied, an approximate 95 % confidence interval for the solar eccentricity is found by formula (2.9) as $\langle 2;4,35,25\ ,\ 2;4,35,34\rangle$. However, it turns out that for all eccentricity values in this interval a cluster of negative errors with respect to the recomputation occurs for $\bar{a} = 8$ to $\bar{a} = 15$. Apparently, the tabular errors in this region do not satisfy the condition of independence. This is confirmed by the result of the statistical tests indicated in Section 1.2.4.

To obtain a valid 95 % confidence interval, we disregard the tabular values for $\bar{a} = 1$ to $\bar{a} = 15$ and compute the weighted estimator $\widehat{e}$ according to

$$\widehat{e} = \frac{1}{W} \sum_{\bar{a}=16}^{179} w_{\bar{a}} \left(\frac{\sin \bar{a}}{\tan T(\bar{a})} - \cos \bar{a} \right), \qquad (2.93)$$

where $w_{\bar{a}} = \left(\dfrac{\sin \bar{a}}{\theta_0^2 + 2\theta_0 \cos \bar{a} + 1} \right)^2$ and $W = \sum_{\bar{a}=16}^{179} w_{\bar{a}}$. (2.93) yields $\widehat{e} = 2;4,35,30,12$ and the corresponding approximate 95 % confidence interval for the eccentricity is given by $\langle\, 2;4,35,26\, ,\ 2;4,35,35\, \rangle$. The tabular errors for $\bar{a} = 16, 17, 18, \ldots, 179$ do pass the test for

[92] See later in this section.

[93] The solar eccentricity e was the actual parameter occurring in the computation of the solar equation by Islamic astronomers. Therefore the value of e underlying the table can be expected to be a round number.

$\bar{a}$	$T(\bar{a})$	$\bar{a}$	$T(\bar{a})$	$\bar{a}$	$T(\bar{a})$	$\bar{a}$	$T(\bar{a})$	$\bar{a}$	$T(\bar{a})$	$\bar{a}$	$T(\bar{a})$
1	0; 2, 0	31	0;59,30	61	1;42,18	91	1;58,59	121	1;43,48	151	0;59,29
2	0; 4, 0	32	1; 1,15	62	1;43,20	92	1;59, 0	122	1;42,45	152	0;57,37
3	0; 6, 1	33	1; 2,58	63	1;44,20	93	1;58,59	123	1;41,40	153	0;55,44
4	0; 8, 1	34	1; 4,40	64	1;45,18	94	1;58,56	124	1;40,33	154	0;53,50
5	0;10, 1	35	1; 6,21	65	1;46,14	95	1;58,50	125	1;39,24	155	0;51,54
6	0;12, 1	36	1; 8, 1	66	1;47, 9	96	1;58,42	126	1;38,14	156	0;49,58
7	0;14, 1	37	1; 9,40	67	1;48, 1	97	1;58,33	127	1;37, 1	157	0;48, 1
8	0;16, 0	38	1;11,18	68	1;48,52	98	1;58,21	128	1;35,46	158	0;46, 3
9	0;17,59	39	1;12,54	69	1;49,41	99	1;58, 6	129	1;34,30	159	0;44, 3
10	0;19,58	40	1;14,29	70	1;50,27	100	1;57,50	130	1;33,12	160	0;42, 3
11	0;21,57	41	1;16, 3	71	1;51,12	101	1;57,31	131	1;31,51	161	0;40, 3
12	0;23,55	42	1;17,36	72	1;51,56	102	1;57,10	132	1;30,30	162	0;38, 1
13	0;25,53	43	1;19, 8	73	1;52,36	103	1;56,48	133	1;29, 6	163	0;35,58
14	0;27,50	44	1;20,38	74	1;53,15	104	1;56,22	134	1;27,40	164	0;33,55
15	0;29,47	45	1;22, 6	75	1;53,52	105	1;55,55	135	1;26,13	165	0;31,51
16	0;31,44	46	1;23,34	76	1;54,27	106	1;55,26	136	1;24,44	166	0;29,47
17	0;33,40	47	1;24,59	77	1;54,59	107	1;54,54	137	1;23,13	167	0;27,42
18	0;35,36	48	1;26,24	78	1;55,30	108	1;54,20	138	1;21,42	168	0;25,36
19	0;37,30	49	1;27,48	79	1;55,59	109	1;53,44	139	1;20, 8	169	0;23,30
20	0;39,25	50	1;29, 8	80	1;56,25	110	1;53, 6	140	1;18,33	170	0;21,23
21	0;41,18	51	1;30,28	81	1;56,50	111	1;52,26	141	1;16,56	171	0;19,16
22	0;43,11	52	1;31,47	82	1;57,12	112	1;51,43	142	1;15,17	172	0;17, 9
23	0;45, 3	53	1;33, 3	83	1;57,33	113	1;50,59	143	1;13,38	173	0;15, 1
24	0;46,54	54	1;34,19	84	1;57,51	114	1;50,12	144	1;11,56	174	0;12,53
25	0;48,45	55	1;35,32	85	1;58, 7	115	1;49,24	145	1;10,13	175	0;10,44
26	0;50,35	56	1;36,45	86	1;58,21	116	1;48,33	146	1; 8,29	176	0; 8,36
27	0;52,24	57	1;37,55	87	1;58,33	117	1;47,40	147	1; 6,44	177	0; 6,27
28	0;54,12	58	1;39, 3	88	1;58,43	118	1;46,45	148	1; 4,56	178	0; 4,18
29	0;55,59	59	1;40,10	89	1;58,50	119	1;45,48	149	1; 3, 9	179	0; 2, 9
30	0;57,45	60	1;41,15	90	1;58,56	120	1;44,49	150	1; 1,19	180	0; 0, 0

Table 2.5: Transcription of the solar equation table in the Shāmil Zīj

independence. Furthermore Monte Carlo tests show that (2.9) constitutes an accurate approximate 95 % confidence interval for solar equation tables with this set of arguments and values to seconds. Our result is confirmed by the "least number of errors criterion": the minimum possible number of errors with respect to recomputation is 16. Between $e = 2;4,35,29,29$ and $e = 2;4,35,32,56$ a recomputation yields 16 and 17 errors alternately.

Using historical arguments, I will now make it plausible that the author of the solar equation table in the Shāmil Zīj based his table on the value 2;4,35,30 of the solar eccentricity e. In the Almagest, Ptolemy computed his value 2;29,30 of the eccentricity from observations of solstices and equinoxes. Subsequently he computed the maximum solar equation $q_{\max}$ in a way equivalent to $q_{\max} = \mathrm{arcSin}\, e$.[94] al-Battānī used the same procedure and found $e = 2;4,45$.[95] The maximum value in his solar equation table is 1;59,10,[96] but in the explanatory text it is stated that "the maximum is 1;59 approximately".[97] Apparently this rounded value became the basis of later solar equation tables. To compute these tables it was necessary to calculate e as $e = \mathrm{Sin}\, 1;59,0,\ldots$. Since $\mathrm{Sin}\, 1;59 = 2;4,35,29,51,\ldots$, it is plausible that the author of the solar equation table in the Shāmil Zīj used the round value 2;4,35,30. Compared to other values in the 95 % confidence interval this value makes the computations relatively easy. Furthermore, it seems probable that the author of the table did not consider it necessary to compute the eccentricity with a still higher accuracy, or that he was not able to obtain a value more accurate than 2;4,35,30.

The solar equation table in the Shāmil Zīj shows 17 errors compared to the recomputation for $e = 2;4,35,30$. Seven of these errors are found for $\bar{a} = 1$ to $\bar{a} = 17$. In this region all tabular errors are negative, so it looks as if the author simply truncated the values of a more accurate solar equation table. The other errors with respect to the recomputation are $-1''$ (for $\lambda = 61, 137, 148$ and 159) or $+1''$ (for $\lambda = 43, 49, 56, 72, 76$ and 103). In four cases the error could have been caused by small computational errors, in two cases by a scribal mistake (confusion of و and ر). As in the case of the ascension tables in the Sanjufīnī Zīj (see Section 2.6.1), I have not further investigated the method of computation of the table.

Conclusion. *The solar equation table in the Shāmil Zīj was very probably computed using* $e = 2;4,35,30$. *In particular, the author of the table did not round the eccentricity to* 2;4,35 *or* 2;4,36.

The desired accuracy of $\mathrm{Sin}\, 1°59'$ could be obtained from any correct sine table with three sexagesimal fractional digits, even if linear interpolation between $\mathrm{Sin}\, 1°$ and $\mathrm{Sin}\, 2°$ had to be used. Furthermore, a correct sine table to seconds with arguments 0;2, 0;4, 0;6, ..., 1;58, 2;0, ... would also lead to $\mathrm{Sin}\, 1°59' = 2;4,35,30$. This is a mere coincidence, since the only two possibilities for the number of thirds in the result are 0 and 30. Now the sine table in the Paris Ms. 2528 of the Shāmil Zīj (folios 10^{v}–14^{r}) gives $\mathrm{Sin}\, x$ to seconds for arguments 0;2, 0;4, 0;6, ... However, this table would not give the value 2;4,35,30 for the eccentricity, since the value for $\mathrm{Sin}\, 1°58'$ contains an error of $-2''$ which is not the result

[94] Heiberg 1898–1903, vol. 1, pp. 232–240 (Greek) or Toomer 1984, pp. 153–157 (English translation).
[95] Nallino 1899–1907, vol. 3, p. 66 (Arabic) or vol. 1, p. 44 (Latin translation).
[96] Nallino 1899–1907, vol. 2, p. 81.
[97] Nallino 1899–1907, vol. 3, p. 66, lines 20–21 (Arabic) or vol. 1, p. 44, lines 14–15 (Latin translation).

of a scribal error.[98] The possibility should not be ruled out that a special computation was conducted to determine the eccentricity with an accuracy that could not be obtained from any available sine table.

The solar equation table on folios 45r–47v of the Baghdādī Zīj[99] is actually attributed to Abu'l-Wafāʾ. Apart from a small number of scribal errors, the table is identical to the solar equation table in the Berlin recension of the zīj of Ḥabash al-Ḥāsib.[100] The table has values to sexagesimal thirds, and a 95 % confidence interval for the eccentricity underlying the table is ⟨2;4,35,29,50 , 2;4,35,30,2⟩. Therefore it is possible that this table too was computed using 2;4,35,30. Nevertheless, a comparison clearly shows that the solar equation table in the Shāmil Zīj is not a rounded version of the table in the Baghdādī Zīj or otherwise related to it, and therefore it is not possible to attribute the Shāmil table to Abu'l-Wafāʾ.

2.6.3 A Table for the True Solar Longitude in the Jāmiʿ Zīj[101]

The *Jāmiʿ Zīj* was written around the year 970 by Kushyār ibn Labbān ibn Bāshahrī Abu'l-Ḥasan al-Jīlī, who worked in Baghdad.[102] The zīj is extant in four manuscripts, namely Istanbul Fatih 3418, Berlin Ahlwardt 5751, Leiden Or. 8 (1054), and Cairo DM 188/2. The Cairo manuscript contains only an incomplete set of tables. In the other three manuscripts we find collections of approximately 50 tables together with explanatory text. From the given tables of contents and from the coherence of the material in these three manuscripts, it can be concluded that both tables and explanatory text were part of the original zīj written by Kushyār. In another of his works, *The Book of the Astrolabe*, Kushyār mentions that he wrote two different zījes, the *Jāmiʿ Zīj* and the *Bāligh Zīj*. This fact might explain various differences between the tables in the four manuscripts.

At the end of the Berlin and Leiden manuscripts of the *Jāmiʿ Zīj* we find a large number of tables which apparently were not part of Kushyār's original work. In many cases these tables display functions which can also be found in the main set of tables. In the Berlin manuscript, a number of planetary equation tables are attributed to Ibn al-Aʿlam (c. 960), some other tables to Abū Maʿshar (Albumasar, c. 850). In the Leiden manuscript, a set of planetary equation tables is taken from the *Fākhir Zīj* by al-Nasawī (c. 1030), some other tables mention al-Bīrūnī (c. 1000) as their author. However, most of the appended tables in both manuscripts are not attributed.

One of the tables in the manuscript Berlin Ahlwardt 5751 which is not part of the main set of tables, occurs on pages 178–179. The first half of this table is entitled "Table of the Solar Equation", the second half "Table of the Equation of the Mean Solar Position". The

[98]The same holds for the sine tables in the other three manuscripts that I inspected.

[99]See footnote 81 and especially Chapter 4, Section 4.1.1.

[100]Berlin Ahlwardt Ms. 5750, folio 30r–31r. This table was analysed in Kennedy & Salam 1967, pp. 493–494.

[101]Section 2.6.3 will be published in *Ad Radices. Festschrift zum 50jährigen Bestehen des Instituts für Geschichte der Naturwissenschaften in Frankfurt am Main*, to be published by Franz Steiner (Stuttgart).

[102]More information about Kushyār ibn Labbān and about the manuscripts of the *Jāmiʿ Zīj* can be found in Section 4.1.2 of this thesis.

argument is the mean solar position and tabular values are displayed in zodiacal signs, degrees, minutes and seconds for every degree of the ecliptic. No further information about the tabulated function or the author of the table is found. In the remainder of this section we will unravel the mathematical structure of this table and will determine the values of the underlying parameters. All through this section the table will be referred to as "the true solar longitude table in the *Jāmiʿ Zīj*".

An inspection of the tabular values reveals that what has been tabulated is the true solar longitude λ as a function of the mean solar longitude $\bar{\lambda}$. In fact, the difference between tabular value and argument is roughly a sinusoidal function which never exceeds 2 in absolute value, is negative from 23° Gemini to 22° Sagittarius and positive otherwise. Thus we expect that the tabulated function will be

$$\lambda(\bar{\lambda}) = \bar{\lambda} - \bar{q}(\bar{\lambda}) = \bar{\lambda} - \arctan\left(\frac{e\sin(\bar{\lambda}-\lambda_A)}{60+e\cos(\bar{\lambda}-\lambda_A)}\right), \tag{2.94}$$

where $\bar{q}$ denotes the solar equation as a function of the mean solar longitude, e is the solar eccentricity and λ_A the solar apogee.[103]

Table 2.6 displays some of the tabular values $T(\bar{\lambda})$ and their first order differences $D^{(1)}(\bar{\lambda}) \stackrel{\text{def}}{=} T(\bar{\lambda}+1) - T(\bar{\lambda})$. It can be seen at once that linear interpolation within intervals of 5 degrees of the argument was applied. Moreover, it seems probable that so-called "distributed linear interpolation" was used: the tabular differences are distributed over the intervals of 5 degrees in such a way that they are increasing or decreasing over as long as possible stretches of the argument.[104] By means of the irregularities in the first order differences, a number of obvious scribal errors in the tabular values can be corrected. For example, the irregularities in the differences for arguments 340 to 346 can be removed by correcting $T(341)$ to 342;57,13 (نو → نز , كج→يج), $T(344)$ to 345;58,25 (له → كه) and $T(346)$ to 347;58,58 (نح→نخ). The correction $T(339) = 340;56,15$ (يو→يه), which restores the distributed linear interpolation pattern for arguments 335 to 340, is somewhat less plausible. All corrections made in this way are listed in the Apparatus in Section 2.6.4. For the analysis of the true solar longitude table we will only make use of the nodes (i.e. the tabular values for multiples of 5°).

To the corrected tabular values for multiples of 5 degrees I applied a least squares estimation assuming that the tabulated function is given by equation (2.94). The results were as follows:

parameter	*95 % confidence interval*
solar eccentricity	⟨ 2; 4, 4 , 2; 5,11⟩
solar apogee	⟨82;25, 6 , 82;55,57⟩

Even though the 95 % confidence intervals contain historically plausible values of the underlying parameters, we must conclude both from the minimum obtainable standard

[103] See also Section 1.3 of this thesis.

[104] See Section 1.1.5 for a more extensive explanation of "distributed linear interpolation". After the obvious scribal errors indicated below have been corrected, 57 out of the 72 intervals of 5 degrees are in agreement with the assumption that distributed linear interpolation was used.

$\bar{\lambda}$	$T(\bar{\lambda})$	$D^{(1)}(\bar{\lambda})$	$\bar{\lambda}$	$T(\bar{\lambda})$	$D^{(1)}(\bar{\lambda})$	$\bar{\lambda}$	$T(\bar{\lambda})$	$D^{(1)}(\bar{\lambda})$
300	301;11,24	1; 1,37	320	321;40, 3	1; 1,10	340	341;56,49	1; 0, 4
301	302;13, 1	1; 1,37	321	322;41,13	1; 1, 9	341	342;56,53	1; 0,44
302	303;14,38	1; 1,37	322	323;42,22	1; 1, 9	342	343;57,37	1; 0,24
303	304;16,15	1; 1,36	323	324;43,31	1; 1, 9	343	344;58, 1	1; 0,34
304	305;17,51	1; 1,36	324	325;44,40	1; 1, 9	344	345;58,35	1; 0,13
305	306;19,27	1; 1,32	325	326;45,49	1; 0,51	345	346;58,48	0;59,30
306	307;20,59	1; 1,32	326	327;46,40	1; 0,51	346	347;58,18	1; 0,50
307	308;22,31	1; 1,32	327	328;47,31	1; 0,50	347	348;59, 8	1; 0,10
308	309;24, 3	1; 1,32	328	329;48,21	1; 0,50	348	349;59,18	1; 0,20
309	310;25,35	1; 1,32	329	330;49,11	1; 0,50	349	350;59,38	1; 0,10
310	311;27, 7	1; 1,18	330	331;50, 1	1; 0,47	350	351;59,48	1; 0, 0
311	312;28,25	1; 1,18	331	332;50,48	1; 0,47	351	352;59,48	1; 0, 0
312	313;29,43	1; 1,18	332	333;51,35	1; 0,47	352	353;59,48	1; 0, 0
313	314;31, 1	1; 1,18	333	334;52,22	1; 0,47	353	354;59,48	1; 0, 1
314	315;32,19	1; 1,18	334	335;53, 9	1; 0,47	354	355;59,49	1; 0, 0
315	316;33,37	1; 1,18	335	336;53,56	1; 0,35	355	356;59,49	0;59,49
316	317;34,55	1; 1,17	336	337;54,31	1; 0,35	356	357;59,38	0;59,49
317	318;36,12	1; 1,17	337	338;55, 6	1; 0,35	357	358;59,27	0;59,49
318	319;37,29	1; 1,17	338	339;55,41	1; 0,35	358	359;59,16	0;59,49
319	320;38,46	1; 1,17	339	340;56,16	1; 0,33	359	360;59, 5	0;59,49
						360	361;58,54	

Table 2.6: Tabular differences of the True Solar Longitude table in Berlin Ahlwardt 5751

deviation of $1'38''$ and from the sinusoidal error pattern in recomputations for parameter values within the confidence intervals, that the table was not computed according to equation (2.94).[105] Therefore we will make use of approximated Fourier coefficients to obtain more information about the tabulated function and to estimate the underlying value of the solar apogee as described in Section 2.3.

Let $T_q(\bar{\lambda}) \stackrel{\text{def}}{=} \bar{\lambda} - T(\bar{\lambda})$ denote the solar equation table that can be reconstructed from the true solar longitude values in the *Jāmiʿ Zīj* by subtracting them from the mean solar longitude. $T_q(\bar{\lambda})$, $\bar{\lambda} = 0, 5, 10, \ldots, 355$ are then tabular values for a function f which satisfies $f(x) = g(x - \lambda_A)$ for every x, where g is an odd function with period $360°$.[106] Since the number of tabular values n in this case is 72, which is a multiple of 4, we can use the theory of the Fourier estimator developed in Section 2.3 without any modification.

[105] Note that for a correct table with values to seconds, the minimum obtainable standard deviation is approximately $17'''$.

[106] The solar equation always has this property, regardless of which of the three common methods was used for its computation (cf. Kennedy 1977 and Section 1.3 of this thesis). Furthermore it can be checked directly that the values $T_q(\bar{\lambda})$ have this property at least approximately.

k	$\widehat{a}_k$	$\widehat{b}_k$
0	−0.0000501543	
1	−1.9684899033	0.2530317106
2	−0.0002099123	−0.0004847958
3	−0.0131708496	0.0049899528
4	−0.0001228737	−0.0003738902
5	0.0004548023	−0.0002402755
6	0.0001358618	0.0002674402
7	0.0006174986	−0.0000334333
8	0.0002106039	0.0005374500
9	−0.0001912837	0.0002584406
10	−0.0000278764	0.0003071745

Table 2.7: Fourier coefficients of the reconstructed solar equation

Table 2.7 displays the approximated Fourier coefficients $\widehat{a}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(5i)\cos 5ik$ for $k = 0, 1, 2, \ldots, 10$ and $\widehat{b}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(5i)\sin 5ik$ for $k = 1, 2, 3, \ldots, 10$.[107] We can note the following:

- The Fourier coefficients converge rapidly, as can be seen from $\widehat{a}_1, \widehat{a}_3, \widehat{a}_5$ and $\widehat{b}_1, \widehat{b}_3, \widehat{b}_5$. Starting from $k = 4$ the coefficients seem to have random values between $-6.2 \cdot 10^{-4}$ and $+3.8 \cdot 10^{-4}$, which are probably the result of rounding errors (and possibly scribal or computational errors) in the tabular values. The same holds for the coefficient $\widehat{a}_0$.
- The coefficients $\widehat{a}_2$ and $\widehat{a}_4$ are significantly smaller than $\widehat{a}_1$ and $\widehat{a}_3$. Similarly, $\widehat{b}_2$ and $\widehat{b}_4$ are significantly smaller than $\widehat{b}_1$ and $\widehat{b}_3$. We conclude that the odd, periodic function g as introduced above also satisfies the symmetry $g(180 - x) = g(x)$ for every x.[108]

We can now compare tabular values $T_q(5i)$ and $T_q(5i + 180)$ to see whether the symmetry $f(x) = -f(x+180)$, which follows from $g(x) = -g(x)$ combined with $g(180-x) = g(x)$, is in fact present in the reconstructed solar equation table. Furthermore, we can compare the tabular values with approximated functional values $f_K(x)$ as defined in formula (2.72):

$$f_K(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\widehat{a}_0 + \sum_{k=1}^{K} \left(\widehat{a}_k \cos kx + \widehat{b}_k \sin kx\right) \tag{2.95}$$

Since all approximated Fourier coefficients starting from $k = 4$ are of the same order of magnitude, we will in this case choose $K = 3$. Table 2.8 shows the reconstructed solar equation values $T_q(5i)$ together with the differences (in seconds) between these values

[107] The coefficients for $k > 10$ show the same behaviour as the coefficients for $k = 4$ to 10.

[108] Cf. Section 2.3.3. If the symmetry $T_q(x) = T_q(x+180)$ holds for all tabular values, the approximated Fourier coefficients $\widehat{a}_k$ and $\widehat{b}_k$ for even k would actually be zero. The fact that they are small but non-zero is a result of scribal or computational errors that will be discovered (and partially corrected) below.

$\bar{\lambda}$	$T_q(\bar{\lambda})$	$T_q(\bar{\lambda}) - f_3(\bar{\lambda})$	$\bar{\lambda}$	$T_q(\bar{\lambda})$	$T_q(\bar{\lambda}) - f_3(\bar{\lambda})$
0	−1;58,54	+1	180	1;58,54	+1
5	−1;56,59	+3	185	1;56,59	−1
10	−1;54,10	+4	190	1;54,10	−1
15	−1;50,27	+5	195	1;50,27	−2
* 20	−1;46, 0	−2	* 200	1;46, 0	+6
25	−1;40,27	+7	205	1;40,27	−4
* 30	−1;34,59	−33	210	1;34,18	−4
* 35	−1;27,44	−9	* 215	1;27,44	+13
40	−1;19,58	+8	220	1;19,58	−4
45	−1;11,56	+6	225	1;11,54	−5
50	−1; 3,24	+5	230	1; 3,24	−2
55	−0;54,24	+6	235	0;54,25	−2
60	−0;45, 6	+2	240	0;45, 9	+3
65	−0;35,27	+2	245	0;35,27	
70	−0;25,37	−1	250	0;25,37	+2
75	−0;15,33	+1	255	0;15,34	+1
80	−0; 5,25		260	0; 5,26	+1
85	0; 4,44	−1	265	−0; 4,44	
90	0;14,52	−2	270	−0;14,52	
95	0;24,56		275	−0;24,55	
100	0;34,48	−2	280	−0;34,48	−1
105	0;44,27	−2	285	−0;44,29	−3
110	0;53,51		290	−0;53,49	−1
115	1; 2,49	−3	295	−1; 2,49	−1
120	1;11,23	−4	300	−1;11,24	−1
125	1;19,27	−5	305	−1;19,27	+1
* 130	1;27, 7	+4	* 310	−1;27, 7	−8
135	1;33,52	−4	* 315	−1;33,37	+16
140	1;40, 3	−4	320	−1;40, 3	+1
* 145	1;46, 1	+27	* 325	−1;45,49	−18
150	1;50,10	−2	* 330	−1;50, 1	+9
155	1;53,56	−2	335	−1;53,56	
160	1;56,50		340	−1;56,49	+1
165	1;58,47		345	−1;58,48	−1
* 170	1;59,36	−11	350	−1;59,48	−1
175	1;59,49		355	−1;59,49	+1

Table 2.8: Differences between the reconstructed solar equation and an approximation

and the approximated functional values $f_3(5i)$. It can be noted that the general error pattern is regular, but that there are many outliers, in particular for arguments 20, 30, 35, 130, 145, 170, 200, 215, 310, 315, 325 and 330 (indicated in the table with an asterisk). The tabular values show the expected symmetry $T_q(5i) = -T_q(5i+180)$ in 20 out of 36 cases. For the pairs of arguments 30/210, 135/315, 145/325, 150/330 and 170/350, $T_q(5i)$ differs by more than 8 seconds from $-T_q(5i+180)$. For eleven other pairs we find a small deviation from the expected symmetry.

It turns out that we can find plausible corrections for four of the outliers. The following list gives the corrected solar equation values, the errors (between brackets) and the corrections in Arabic *abjad* notation.[109]

$T(\ 30)$	=	$-1;34,19$	$(-40'')$	نط → يط
$T(170)$	=	$1;59,47$	$(-11'')$	لو → مز
$T(315)$	=	$-1;33,52$	$(+15'')$	لز → نب
$T(330)$	=	$-1;50,11$	$(+10'')$	ا → يا

Note that in all four cases the correction (approximately) restores both the surrounding error pattern and the symmetry $T_q(5i) = -T_q(5i+180)$. Furthermore all four errors are plausible scribal errors. The other eight outliers occur in pairs $T_q(5i)/T_q(5i+180)$ and can therefore not be corrected on the basis of the symmetry of the reconstructed solar equation. Since no plausible corrections on the basis of possible scribal mistakes can be given either, we will leave these outliers unchanged for the time being.

Since the solar equation is almost linear in the neighbourhood of its zeros, inverse linear interpolation between $T_q(80)$ and $T_q(85)$ or $T_q(260)$ and $T_q(265)$ ordinarily leads to reasonably accurate apogee values (in this case the results are $\lambda_A \approx 82;40,6$ and $\lambda_A \approx 82;40,20$ respectively). However, because of the large number of errors in the reconstructed solar equation values, it seems advisable to compute a more accurate estimate and a confidence interval based on all tabular values by means of the Fourier estimator developed in Section 2.3. Thus we calculate the approximate Fourier coefficients anew for the corrected solar equation values and arrive at an estimate $\widehat{\lambda_A} \stackrel{\text{def}}{=} -\arctan(\widehat{a}_1/\widehat{b}_1) = 82;40,1$ for the apogee. Using

$$\sigma^2 \approx \frac{1}{n-1}\sum_{i=1}^{n}\left(T_q(5i) - f_3(5i)\right)^2, \tag{2.96}$$

with f_3 as in equation (2.95), we find that the standard deviation σ of the tabular errors is approximately equal to $4''51'''$. From formula (2.58) it follows that $\langle\, 82;38,56\,,\ 82;41,6\,\rangle$ is an approximate 95 % confidence interval for the solar apogee. However, since the condition that the tabular errors have equal variances is not satisfied because of the outliers, the interval probably is a "slightly less than 95 %" confidence interval.

Seeing that none of the remaining outliers occurs for a multiple of 15°, we can obtain a better estimate and a smaller approximate 95 % confidence interval for the solar apogee by applying the Fourier estimator to the set of solar equation values $T_q(15i)$, $i = 1, 2, 3, \ldots, 24$. It turns out that we can then use $K = 5$ in our approximation (2.95)

[109] See 1.1.1 for an explanation of the *abjad* notation.

of the functional values, since $\widehat{a}_5$ and $\widehat{b}_5$ are significantly larger than the approximated Fourier coefficients with larger indices. The resulting estimate for the solar apogee is $\widehat{\lambda_A} = 82;40,6$, the approximated standard deviation of the tabular errors $52'''47^{\text{iv}}$. An approximate 95 % confidence interval for the apogee is found as $\langle\, 82;39,46\,,\, 82;40,26\,\rangle$.[110] Since the differences $T_q(15i) - f_5(15i)$ in fact do not show any outliers, we conclude that the table for true solar longitude in the *Jāmiʿ Zīj* was very probably computed on the basis of the round solar apogee value $\lambda_A = 82;40$. Below I will explain why this value is also historically plausible.

Because of the symmetry $g(180 - x) = g(x)$ discovered above, we know that the maximum solar equation is assumed for $\lambda_A + 90°$, the minimum solar equation for $\lambda_A + 270°$. By means of third order interpolation between the reconstructed values $T_q(165)$, $T_q(170)$, $T_q(175)$ and $T_q(180)$, we find that the maximum equation is approximately equal to 1;59,55,13. Similarly, we find that the minimum equation is close to −1;59,55,41. We conclude that the reconstructed solar equation is probably based on a value of $q_{\max}$ close to 1;59,55 or 1;59,56 (or, equivalently, on a value of the solar ecccentricity e in the neighbourhood of 2;5,34). These values are not common in Islamic astronomical handbooks. However, a maximum solar equation value $q_{\max} = 1;59,56$ is attested in a solar equation table in the *Ashrafī Zīj* which is attributed to Yaḥyā ibn Abī Manṣūr, one of the astronomers who worked at the court in Baghdad around the year 830.[111] Kennedy found that this solar equation table, which has the mean solar anomaly as its independent variable, was computed according to the so-called "method of declinations", which is probably of Sasanian or early-Islamic origin.[112] We will investigate whether the same holds for the true solar longitude table in the *Jāmiʿ Zīj*, i.e. whether the tabulated function is

$$f_{q_{\max},\varepsilon,\lambda_A}(x) = q_{\max} \cdot \frac{\arcsin(\sin(x - \lambda_A) \cdot \sin\varepsilon)}{\varepsilon}. \tag{2.97}$$

Note that this function satisfies both symmetry relations $f(x - \lambda_A) = -f(-x - \lambda_A)$ and $f(180 - x - \lambda_A) = f(x - \lambda_A)$ that we have also found in the table.

Disregarding the outliers for arguments 20, 35, 130, 145, 200, 215, 310 and 325, which could not plausibly be corrected, I performed several Least Squares estimations as explained in Section 2.4. Assuming that the true solar longitude table in the *Jāmiʿ Zīj* was computed according to the method of declinations, I found that the minimum obtainable standard deviation of the tabular errors is $60'''$, and I obtained the following approximate 95 % confidence intervals:

[110] For the calculation of the confidence interval I assumed that $T_q(5i) = -T_q(5i + 180)$. Since this is not the case for every i, we have now actually found a "slightly more than 95 %" confidence interval.

[111] See Section 2.6.2, footnote 82 for more information about the *Ashrafī Zīj*. The solar equation table concerned can be found on folio 236ʳ of the manuscript Paris Bibliothèque Nationale Suppl. Persane 1488, the only extant copy of the *Ashrafī Zīj*.

[112] See Kennedy 1977 and Section 1.3 of this thesis.

parameter	*95 % confidence interval*
maximum solar equation	⟨ 1;59,55,18 , 1;59,55,47⟩
obliquity of the ecliptic	⟨23;39,15 , 23;48,21 ⟩
solar apogee	⟨82;39,53 , 82;40,13 ⟩

Since the minimum obtainable standard deviation is much higher if we assume any other plausible method of computation,[113] we conclude that the table was very probably computed according to the method of declinations. The confidence interval for the solar apogee confirms the results that we found before by means of the Fourier estimator, so in fact $\lambda_A = 82;40$. Since the method of declinations is only attested for the Ptolemaic obliquity value,[114] we can assume that $\varepsilon = 23;51$. Fixing these two parameter values, we obtain ⟨1;59,55,27 , 1;59,56,12⟩ as an approximate 95 % confidence interval for the maximum solar equation. Thus we see that the true solar longitude in the *Jāmiʿ Zīj* was probably computed using $q_{\max} = 1;59,56$.

Conclusion. *The table for true solar longitude which is found on pages 178–179 of the manuscript Berlin Ahlwardt 5751 of Kushyār ibn Labbān's Jāmiʿ Zīj was computed according to the so-called "method of declinations" (formula 2.97). The underlying parameter values are* 1;59,56 *for the maximum solar equation,* 23;51 *for the obliquity of the ecliptic and* 82;40 *for the solar apogee.*

I will now argue that all three underlying parameter values are historically plausible and that the true solar longitude table in the *Jāmiʿ Zīj* probably derives from Yaḥyā ibn Abī Manṣūr. We have already seen that 1;59,56 is the maximum solar equation value underlying Yaḥyā's solar equation table in the *Ashrafī Zīj*. Furthermore, we have seen that the method of declinations is only attested with the value 23;51 of the obliquity of the ecliptic. I conjecture that the solar apogee value 82;40 is a rounded version of the value 82;39, which, according to Ibn Yūnus, was observed at Baghdad in the year 214 Hijra by a group of astronomers headed by Yaḥyā ibn Abī Manṣūr.[115] The solar equation tables in Yaḥyā's *Mumtaḥan Zīj* and in the zīj by Ḥabash al-Ḥāsib extant in Istanbul Yeni Cami 784/2, both indicate that the solar apogee is in $82°39'$.[116] The two tables are very probably related, since the first 90 values are practically identical. Ḥabash's zīj contains another table based on the same solar equation values, which displays λ_A plus the solar equation. Here the apogee is taken equal to $82°40'$.[117]

We have seen that the true solar longitude table in the *Jāmiʿ Zīj* was computed

[113] We have already seen that the minimum obtainable standard deviation is $1'38''$ if we assume that the correct formula $\bar{q}(\bar{\lambda}) = \arctan\left(\dfrac{e\sin(\bar{\lambda}-\lambda_A)}{60+e\cos(\bar{\lambda}-\lambda_A)}\right)$ for the solar equation was applied. Assuming the formula $q(\lambda) = \arcsin(\frac{e}{60}\sin(\lambda-\lambda_A))$ for the solar equation as a function of the true solar longitude or the so-called "method of sines" $q(\lambda) = q_{\max}\sin(\lambda-\lambda_A)$, the minimum obtainable standard deviation is $38''$.

[114] See Kennedy & Muruwwa 1958, p. 118; Kennedy 1977 and Suter 1914, pp. 132–137.

[115] See Caussin de Perceval 1804, p. 56 (p. 40 in the separatum).

[116] For the table in the *Mumtaḥan Zīj*, see Escorial Ms. árabe 927, folio 15ʳ or Yaḥyā ibn Abī Manṣūr, p. 28. For the table in Ḥabash's zīj, see Istanbul Yeni Cami 784/2, folios 90ʳ–91ʳ and Debarnot 1987, pp. 41–42. Both tables were analysed in Salam & Kennedy 1967, pp. 494–495. Note that the solar equation table in the *Mumtaḥan Zīj* is completely different from the table in the *Ashrafī Zīj* attributed

λ	$T_q(\lambda)$	error	λ	$T_q(\lambda)$	error
0	−1;58,54		180	1;58,54	
5	−1;56,59	+1	185	1;56,59	−1
10	−1;54,10		190	1;54,10	
15	−1;50,27		195	1;50,27	
* 20	−1;46, 0	−9	* 200	1;46, 0	+9
25	−1;40,27	+1	205	1;40,27	−1
30	−1;34,19	−1	210	1;34,18	
* 35	−1;27,44	−17	* 215	1;27,44	+17
40	−1;19,58	+1	220	1;19,58	−1
45	−1;11,56		225	1;11,54	−2
50	−1; 3,24	−1	230	1; 3,24	+1
55	−0;54,24	+1	235	0;54,25	
60	−0;45, 6	−1	240	0;45, 9	+4
65	−0;35,27		245	0;35,27	
70	−0;25,37	−2	250	0;25,37	+2
75	−0;15,33		255	0;15,34	+1
80	−0; 5,25		260	0; 5,26	+1
85	0; 4,44	−1	265	−0; 4,44	+1
90	0;14,52	−1	270	−0;14,52	+1
95	0;24,56	+1	275	−0;24,55	
100	0;34,48		280	−0;34,48	
105	0;44,27		285	−0;44,29	−2
110	0;53,51	+3	290	−0;53,49	−1
115	1; 2,49	+1	295	−1; 2,49	−1
120	1;11,23		300	−1;11,24	−1
125	1;19,27	−1	305	−1;19,27	+1
* 130	1;27, 7	+8	* 310	−1;27, 7	−8
135	1;33,52		315	−1;33,52	
140	1;40, 3	−1	320	−1;40, 3	+1
* 145	1;46, 1	+30	* 325	−1;45,49	−18
150	1;50,10		330	−1;50,11	−1
155	1;53,56	−1	335	−1;53,56	+1
160	1;56,50		340	−1;56,49	+1
165	1;58,47	−1	345	−1;58,48	
170	1;59,47	−1	350	−1;59,48	
175	1;59,49	−1	355	−1;59,49	+1

Table 2.9: Final recomputation of the reconstructed solar equation

by means of so-called "distributed linear interpolation". The extant recension of the *Mumtaḥan Zīj* contains a table for the normed right ascension, which is based on obliquity 23;51 and involves the same type of interpolation.[118] Although this table may simply have been copied from Ptolemy's *Handy Tables*,[119] it seems probable that it was an original part of Yaḥyā ibn Abī Manṣūr's zīj, and hence that Yaḥyā was familiar with distributed linear interpolation.

Finally, it can be noted that among the appended tables in the manuscript Berlin Ahlwardt 5751, we find two tables displaying mean planetary positions at two different epochs according to four astronomers, one of them being Yaḥyā ibn Abī Manṣūr.[120] Thus the compiler of the manuscript apparently had access to Yaḥyā's zīj.

We conclude that there is sufficient reason to believe that the true solar longitude table analysed here, like the solar equation table on folio 236^{r} of the *Ashrafī Zīj*, originates from Yaḥyā ibn Abī Manṣūr. It seems possible that Yaḥyā's solar equation table as found in the *Ashrafī Zīj*, was originally contained in the *Mumtaḥan Zīj*, but was later considered unsatisfactory because of its symmetry (and possibly because of its uncommon value of the maximum equation). Thus we can imagine how in a later recension, like the one that we find in the manuscript Escorial árabe 927, it was replaced, possibly by Ḥabash's table for the solar equation.

Table 2.9 displays my final recomputation of the solar equation reconstructed from the true solar longitude table in the *Jāmiʿ Zīj*. The second and fifth columns contain the reconstructed solar equation values, the third and sixth columns the differences (in seconds) between these values and a recomputation according to formula (2.97) using the parameter values found above. Apart from the eight outliers (which are again indicated by an asterisk), the number of differences is 40 out of 64 tabular values, the standard deviation of the differences is $1''5'''$.

It seems probable that the true solar longitude table in the *Jāmiʿ Zīj* was computed by means of interpolation in a solar equation table like the one in the *Ashrafī Zīj*. In fact, if linear interpolation were used, the remaining eight outliers in our table could be explained from only two erroneous solar equation values. To see this, we denote the values for the method of declination that were used for the linear interpolation by $q_\delta(\bar{a})$, where $\bar{a} = 1, 2, 3, \ldots, 90$ is the mean solar anomaly. Remembering that $q_\delta(-\bar{a}) = -q_\delta(\bar{a})$ and $q_\delta(180 - \bar{a}) = q_\delta(\bar{a})$, it follows that $T_q(35)$ and hence $-T_q(215)$ were calculated as $-\frac{1}{3}q_\delta(47) - \frac{2}{3}q_\delta(48)$, and $T_q(130)$ and $-T_q(310)$ as $\frac{2}{3}q_\delta(47) + \frac{1}{3}q_\delta(48)$. The solar equation values $q_\delta(47) = 1;26,30$ (equal to the value given in the *Ashrafī Zīj*) and $q_\delta(48) = 1;28,21$ (computational error for 1;27,56 ?) thus precisely reproduce the four outliers for arguments 35, 130, 215 and 310. In the same way three of the remaining four outliers can be

to Yaḥyā.

117 See Istanbul Yeni Cami 784/2, folios 200^{v}–203^{r} and Debarnot 1987, p. 58.

118 Escorial Ms. árabe 927, folios 48^{v}–49^{r} or Yaḥyā ibn Abī Manṣūr, pp. 93–94. See Section 1.3 of this thesis for more information about the normed right ascension.

119 The normed right ascension table in the *Handy Tables* can for instance be found in the manuscript Leiden BPG 78, folios 75^{r}–76^{v}. The table is transcribed in Stahlman 1959, pp. 206–209. The normed right ascension in the *Mumtaḥan Zīj* is practically identical to the table in the *Handy Tables*.

120 See Berlin Ahlwardt 5751, pp. 160–161.

explained if we assume an erroneous value $q_\delta(62) = 1;45,38$ (possible scribal error (يج → لح) for the *Ashrafī* value 1;45,13).

I recomputed the true solar longitude table in the *Jāmiʿ Zīj* by using linear interpolation in Yaḥyā ibn Abī Manṣūr' solar equation table in the *Ashrafī Zīj*. Disregarding the eight outliers, I found 28 differences in 64 values (as compared to 40 differences for the precise recomputation); the standard deviation of the differences was $1''3'''$. This result is not good enough to conclude that in fact linear interpolation in the *Ashrafī* table was applied.

2.6.4 Apparatus

The right ascension table on folio 39r of the Sanjufīnī Zīj

Scribal error (the correction is given between brackets):

$$\lambda' = 66 \quad T(\lambda') = 66;48 \qquad (67^\circ)$$

Errors with respect to the recomputation for $\varepsilon = 23;35$ (the recomputed values are given in brackets):

$\lambda' = 6$	$T(\lambda') = 6;32$	$(33')$	$\lambda' = 46$	$T(\lambda') = 48;30$	$(29')$
16	17;23	$(22')$	54	56;20	$(21')$
17	18;28	$(27')$	55	57;18	$(19')$
20	21;39	$(40')$	56	58;16	$(17')$
21	22;43	$(44')$			

The oblique ascension table on folio 38v of the Sanjufīnī Zīj

The tabular values for arguments 95, 130, and 149 are more or less illegible. Nevertheless they were reliably restored, since the number of minutes could be identified in each case. Scribal error:

$$\lambda = 163 \quad T(\lambda) = 119;\ 2 \qquad (159^\circ)$$

Right ascension extracted from the oblique ascension table on folio 38v of the Sanjufīnī Zīj

Differences larger than 0;0,30 between the extracted values and recomputed values for $\varepsilon = 23;32,30$ (the recomputed values to seconds are given in brackets):

$\lambda = 5$	$T_\alpha(\lambda) = 4;36,\ 0$	(4;35, 9)	$\lambda = 49$	$T_\alpha(\lambda) = 46;32,\ 0$	(46;31,22)
7	6;26, 0	(6;25,21)	55	52;37, 0	(52;37,42)
10	9;11,30	(9;10,57)			

Solar equation table in the Shāmil Zīj

Scribal errors (the corrections in brackets are based entirely on the interpolation pattern, which is very reliable; note that between $\bar{a} = 105$ and $\bar{a} = 108$ several digits were shifted upwards):

$\bar{a} = 21$	$T(\bar{a}) = 0;41,\ 0$	$(18'')$	$\bar{a} = 114$	$T(\bar{a}) = 1;55,52$	$(50'12'')$
27	0;52,29	$(24'')$	121	1;43,43	$(48'')$
38	1;11,48	$(18'')$	128	1;25,46	$(35')$
47	1;24,19	$(59'')$	129	1;34,36	$(30'')$
55	1;35,37	$(32'')$	146	1;10,29	$(8')$
66	1;47,20	$(9'')$	149	1; 2, 9	$(3')$
105	1;55,27	$(55'')$	155	0;51,14	$(54'')$
106	1;55,24	$(26'')$	163	0;35,18	$(58'')$
107	1;55,20	$(54'54'')$	169	0;28,30	$(23')$
108	1;53,44	$(54'20'')$	171	0;19,56	$(16'')$

Errors with respect to the recomputation for $e = 2;4,35,30$ (digits of the recomputed values differing from the corresponding digits in the Shāmil Zīj are given between brackets; the errors for $\bar{a} = 148$ and $\bar{a} = 159$ are *not* scribal errors):

$\bar{a} = 2$	$T(\bar{a}) = 0;\ 4,\ 0$	$(1'')$	$\bar{a} = 56$	$T(\bar{a}) = 1;36,45$	$(44'')$
8	0;16, 0	$(1'')$	61	1;42,18	$(19'')$
9	0;17,59	$(18'0'')$	72	1;51,56	$(55'')$
10	0;19,58	$(59'')$	76	1;54,27	$(26'')$
12	0;23,55	$(56'')$	103	1;56,48	$(47'')$
14	0;27,50	$(51'')$	137	1;23,13	$(14'')$
15	0;29,47	$(48'')$	148	1; 4,56	$(57'')$
43	1;19, 8	$(7'')$	159	0;44, 3	$(4'')$
49	1;27,48	$(47'')$			

True solar longitude table in Kushyār ibn Labbān's Jāmiʿ Zīj
Scribal errors corrected on the basis of the interpolation pattern (the corrections are given between brackets; possible errors of $1''$ for arguments 204, 206 and 212 were not corrected):

$\bar{\lambda} = 21$	$T(\bar{\lambda}) = 22;44,33$	$(53'')$	$\bar{\lambda} = 136$	$T(\bar{\lambda}) = 134;24,13$	$(53'')$
33	34;30,18	$(38'')$	146	144;14,59	$(13')$
47	48; ,22	$(32'')$	147	145;12,59	$(19'')$
57	57;50,40	$(42'')$	181	179; 1,21	$(29'')$
78	78; ,39	$(29'')$	279	279;32,30	$(50'')$
104	103;17,18	$(28'')$	298	299; 7,18	$(58'')$
116	116;15,28	$(55')$	341	342;56,53	$(57'13'')$
117	115;13,45	$(53')$	344	345;58,35	$(25'')$
130	128;32,13	$(53'')$	346	347;58,18	$(58'')$

2.A Appendix

2.A.1 Optimal Weights

It is a well-known fact from Sampling Theory that the variance of a weighted sum of independent estimators is minimized when the weights are chosen inversely proportionate to the variances of the separate estimators. This fact can be proven as follows.

Let $\widehat{\theta} \stackrel{\text{def}}{=} \sum_x w_x \widehat{\theta}_x$ with $\sum_x w_x = 1$ be a weighted sum of the independent estimators $\widehat{\theta}_x$ and assume that $\operatorname{Var}\widehat{\theta} = \sum_x w_x^2 \operatorname{Var}\widehat{\theta}_x$ is minimized under the condition $\sum_x w_x - 1 = 0$ for values $\overline{w_x}$ of the weights w_x. According to the multiplier theorem of Lagrange there exist constants λ_0 and λ_1 such that for every $\widetilde{x}$

$$\frac{\partial}{\partial w_{\widetilde{x}}} \left(\lambda_0 \cdot \sum_x w_x^2 \operatorname{Var}\widehat{\theta}_x + \lambda_1 \cdot \left(\sum_x w_x - 1 \right) \right) \Bigg|_{w_{\widetilde{x}} = \overline{w_{\widetilde{x}}}} = 0. \tag{2.98}$$

We find that, for every $\widetilde{x}$, $2\lambda_0 \overline{w_{\widetilde{x}}} \operatorname{Var}\widehat{\theta}_{\widetilde{x}} + \lambda_1 = 0$ and thus $\overline{w_{\widetilde{x}}} = \dfrac{-\lambda_1}{2\lambda_0} \cdot \dfrac{1}{\operatorname{Var}\widehat{\theta}_{\widetilde{x}}}$. Thus the optimal weights $\overline{w_x}$ are inversely proportionate to $\operatorname{Var}\widehat{\theta}_x$.

2.A.2 The Distribution of the Weighted Estimator

In Loève 1977–1978 the following version of the Central Limit Theorem for the case of bounded variances is proven:[121]

Theorem. Let X_{ni}, $n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots, n$ be independent random variables with probability densities h_{ni}. Assume that the X_{ni} are *uniformly asymptotically negligible*, i.e. that $\lim\limits_{n\to\infty} \max\limits_{i=1,\ldots,n} \Pr(|X_{ni}| \geq \epsilon) = 0$ for every $\epsilon > 0$. Furthermore, assume that the X_{ni} have mean 0 and that $\sum\limits_{i=1}^{n} \operatorname{Var} X_{ni} = 1$ for every n. Then we have:
The distribution of $\sum\limits_{i=1}^{n} X_{ni}$ *converges to the standard normal distribution for* $n \to \infty$ *and* $\lim\limits_{n\to\infty} \max\limits_{i=1,\ldots,n} \operatorname{Var} X_{ni} = 0$ **if, and only if,** $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \int\limits_{|y|\geq\epsilon} y^2 h_{ni}(y)\, dy = 0$ *for every* $\epsilon > 0$.

We will apply this theorem to the weighted estimator as defined in Section 2.2.1. Let $T(x_{ni})$, $n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots, n$ be tabular values for a function f_θ based on a parameter θ. Let g_{ni} be functions such that $g_{ni}(f_\theta(x_{ni})) = \theta$ for every n, i and θ. Then $\widehat{\theta}_{ni} \stackrel{\text{def}}{=} g_{ni}(T(x_{ni}))$ is an estimator for the parameter θ for every n and i. Assume that the tabular errors e_{ni} defined by $e_{ni} \stackrel{\text{def}}{=} T(x_{ni}) - f_\theta(x_{ni})$ are independent and have common mean 0 and variance σ^2. We will consider the weighted estimator $\widehat{\theta}_n \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \frac{w_{ni}}{W_n} \widehat{\theta}_{ni}$ with optimal weights $w_{ni} \stackrel{\text{def}}{=} \dfrac{1}{\operatorname{Var} \widehat{\theta}_{ni}}$ and $W_n = \sum\limits_{i=1}^{n} w_{ni}$.

Let X_{ni}, $n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots, n$ be defined by $X_{ni} = \dfrac{w_{ni}}{\sqrt{W_n}} (\widehat{\theta}_{ni} - E\widehat{\theta}_{ni})$ with w_{ni} and W_n as above. Since the tabular errors e_{ni} are independent, the separate estimators $\widehat{\theta}_{ni}$ and hence also the random variables X_{ni} are independent. Furthermore, the X_{ni} have mean 0 and we have $\sum\limits_{i=1}^{n} \operatorname{Var} X_{ni} = \sum\limits_{i=1}^{n} \dfrac{w_{ni}^2}{W_n} \operatorname{Var} \widehat{\theta}_{ni} = \sum\limits_{i=1}^{n} \dfrac{w_{ni}}{W_n} = 1$ for every n. We assume that there exists a constant $a > 0$ such that $\operatorname{Var} \widehat{\theta}_{ni} \geq \frac{1}{a}$ (and hence $w_{ni} \leq a$) for all i, and constants $b > 0$ and $0 < \alpha \leq 1$ such that for every n there is at least a fraction α of all values of i smaller than n for which $\operatorname{Var} \widehat{\theta}_{ni} \leq \frac{1}{b}$ (and hence $w_{ni} \geq b$).[122] By applying

[121] See Loève 1977–1978, vol. 1, pp. 300–308.

[122] Cf. the examples for the right ascension and the solar equation in Sections 2.2.2 and 2.2.3. In the case of the right ascension, $\operatorname{Var} \widehat{\theta}_{ni}$ assumes a positive global minimum for $x_{ni} \approx 47\frac{1}{2}^\circ$ and $\operatorname{Var} \widehat{\theta}_{ni} \to \infty$ only if $x_{ni} \downarrow 0°$ or $x_{ni} \uparrow 90°$. In the case of the solar equation, $\operatorname{Var} \widehat{\theta}_{ni}$ assumes a positive global minimum for $x_{ni} \approx 92°$ and $\operatorname{Var} \widehat{\theta}_{ni} \to \infty$ only if $x_{ni} \downarrow 0$ or $x_{ni} \uparrow 180°$. In general, it can be noted that

$$\operatorname{Var} \widehat{\theta}_{ni} = E(\widehat{\theta}_{ni} - E\widehat{\theta}_{ni})^2 = \int\limits_{-\infty}^{+\infty} \left\{ g_{ni}(f_\theta(x_{ni}) + e) - E\widehat{\theta}_{ni} \right\}^2 k_{ni}(e)\, de,$$

where k_{ni} is the density of the tabular error e_{ni}. Since $y \to g_{ni}(y)$ is the inverse function of $\theta \to f_\theta(x_{ni})$, g_{ni} is monotone and $\operatorname{Var} \widehat{\theta}_{ni} > 0$. Note, however, that for functions like $f_\theta^{(1)}(x) = \theta/x$ and $f_\theta^{(2)}(x) = \theta \tan x$ it is possible to construct sequences $\{x_i\}$, $i = 1, 2, 3, \ldots$ such that $\lim\limits_{n\to\infty} \operatorname{Var} \widehat{\theta}_i = 0$ (the respective variances are $\operatorname{Var} \widehat{\theta}_i^{(1)} = x_i^2 \sigma^2$ and $\operatorname{Var} \widehat{\theta}_i^{(2)} = \sigma^2/\tan^2 x_i$, where σ^2 is the variance of the tabular errors). For such

Chebyshev's inequality, it follows that, for every $\epsilon > 0$,

$$\Pr(|X_{ni}| \geq \epsilon) \leq \frac{\operatorname{Var} X_{ni}}{\epsilon^2} = \frac{w_{ni}}{\epsilon^2 W_n} \leq \frac{a}{\alpha b n \epsilon^2} \tag{2.99}$$

for every n and i, implying that the X_{ni} are uniformly asymptotically negligible.

From the version of the Central Limit Theorem presented above we conclude that if

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int\limits_{|y|\geq\epsilon} y^2 h_{ni}(y)\,dy = 0 \qquad \text{for every } \epsilon > 0, \tag{2.100}$$

where h_{ni} is the density of X_{ni}, then the distribution of $\sum_{i=1}^{n} X_{ni}$ converges to the standard normal distribution as $n \to \infty$.

Below I will investigate the linear approximation $\widetilde{\theta}_n = \sum_{i=1}^{n} \frac{\widetilde{w}_{ni}}{\widetilde{W}_n}\widetilde{\theta}_{ni}$ to the weighted estimator $\widehat{\theta}_n$ as defined above. Here $\widetilde{\theta}_{ni} = \theta + g'_{ni} \cdot e_{ni}$ (where g'_{ni} is an abbreviation for $g'_{ni}(f_\theta(x_{ni}))$), $\widetilde{w}_{ni} = \frac{1}{\operatorname{Var}\widetilde{\theta}_{ni}} = \frac{1}{{g'_{ni}}^2\sigma^2}$ and $\widetilde{W}_n = \sum_{i=1}^{n} \widetilde{w}_{ni}$. It will be shown that for $\widetilde{\theta}_n$ the condition (2.100) is satisfied if the tabular errors e_{ni} are uniformly bounded or, more generally, if they have uniformly bounded $2+\delta$ moments for a $\delta > 0$.[123] The same is likely to hold for the weighted estimator $\widehat{\theta}_n$ itself.

Case 1. Uniformly bounded tabular errors. Let $M > 0$ be such that $|e_{ni}| \leq M$ for every n and i. We then have $|\widetilde{\theta}_{ni} - \theta| = |e_{ni} g'_{ni}| \leq M|g'_{ni}|$ for every n and i. Let $\epsilon > 0$ be given and choose $N = \frac{aM^2}{\alpha b \epsilon^2 \sigma^2}$, where σ^2 is the variance of the tabular errors e_{ni}, and a, α and b are as defined above. Let $\widetilde{X}_{ni} = \frac{\widetilde{w}_{ni}}{\sqrt{\widetilde{W}_n}}(\widetilde{\theta}_{ni} - \theta)$. Since $\widetilde{w}_{ni} = \frac{1}{\operatorname{Var}\widetilde{\theta}_{ni}} = \frac{1}{{g'_{ni}}^2\sigma^2}$, it follows that for every $n \geq N$ and for every $i \leq n$

$$\left|\widetilde{X}_{ni}\right| \leq \frac{\widetilde{w}_{ni}}{\sqrt{\widetilde{W}_n}} M g'_{ni} = \frac{\sqrt{\widetilde{w}_{ni}}M}{\sqrt{\widetilde{W}_n}\sigma} \leq \frac{\sqrt{a}M}{\sqrt{\alpha b n}\sigma} \leq \epsilon. \tag{2.101}$$

Consequently, if h_{ni} is the density of $\widetilde{X}_{ni}$, $\int_{|y|\geq\epsilon} y^2 h_{ni}(y)\,dy = 0$ for every $n \geq N$ and $i \leq n$, and the condition (2.100) is satisfied. We conclude that $\widetilde{\theta}_n$ has approximately a normal distribution with mean θ and variance $\widetilde{W}_n^{-1}$. (Note that this also follows from Case 2 below, since uniformly bounded random variables have uniformly bounded finite moments.)

---

functions the condition that all tabular errors have equal variances will generally not be satisfied. Both problems can be solved by excluding parts of the domain around the critical points.

[123] Note that if the tabular errors are normally distributed, $\widetilde{\theta}_n$ will be normally distributed as well, since $\widetilde{\theta}_n = \sum_{i=1}^{n} \frac{w_{ni}}{W_n}\widetilde{\theta}_{ni} = \theta + \sum_{i=1}^{n} \frac{w_{ni}}{W_n} g'_{ni} e_{ni} = \theta + \sum_{i=1}^{n} \frac{\sqrt{w_{ni}}}{W_n \sigma} e_{ni}$.

Case 2. Tabular errors with uniformly bounded $2+\delta$ moments. Let k_{ni}, $n = 1,2,3,\ldots$, $i = 1,2,3,\ldots,n$ denote the densities of the tabular errors e_{ni}, and let $\delta > 0$ and $M > 0$ be such that $\int_{-\infty}^{+\infty} |x|^{2+\delta} k_{ni}(x)\,dx \leq M$ for all n and i. If we write h_{ni} for the density of $\widetilde{X}_{ni}$ given by $\widetilde{X}_{ni} = \dfrac{\widetilde{w}_{ni}}{\sqrt{\widetilde{W}_n}}(\widetilde{\theta}_{ni} - \theta) = \dfrac{\sqrt{\widetilde{w}_{ni}}}{\sqrt{\widetilde{W}_n}\sigma} e_{ni}$, we have, for any given $\epsilon > 0$,

$$
\begin{aligned}
\sum_{i=1}^{n} \int_{|y|\geq\epsilon} y^2 h_{ni}(y)\,dy &\leq \sum_{i=1}^{n} \int_{|y|\geq\epsilon} \epsilon^{-\delta}|y|^{2+\delta} h_{ni}(y)\,dy \\
&\leq \sum_{i=1}^{n} \epsilon^{-\delta} \int_{-\infty}^{+\infty} |y|^{2+\delta} h_{ni}(y)\,dy \\
&\leq \sum_{i=1}^{n} \epsilon^{-\delta}\sigma^{-2-\delta} \left(\frac{\sqrt{\widetilde{w}_{ni}}}{\sqrt{\widetilde{W}_n}}\right)^{2+\delta} \int_{-\infty}^{+\infty} |x|^{2+\delta} k_{ni}(x)\,dx \\
&\leq \sum_{i=1}^{n} \epsilon^{-\delta}\sigma^{-2-\delta} \frac{\widetilde{w}_{ni}}{\widetilde{W}_n} \left(\frac{a}{\alpha b n}\right)^{\frac{1}{2}\delta} M \\
&= \epsilon^{-\delta}\sigma^{-2\delta} \left(\frac{a}{\alpha b n}\right)^{\frac{1}{2}\delta} M, \qquad (2.102)
\end{aligned}
$$

where a, α and b are as defined above. It follows that condition (2.100) is satisfied and hence that $\widetilde{\theta}_n$ has approximately a normal distribution with mean θ and variance $\widetilde{W}_n^{-1}$.

Case 3. Tabular errors not satisfying the condition. It can be seen as follows that for certain distributions of the tabular errors having uniformly bounded variances, the weighted estimator $\widehat{\theta}$ does not converge to a normal distribution. Let f_θ be the constant function $f_\theta(x) = \theta$ for every x.[124] Assume that we have tabular values $T(x_{ni})$ for $f_\theta(x_{ni})$ for arguments x_{ni}, $n = 1,2,3,\ldots$, $i = 1,2,3,\ldots,n$. Then θ can be estimated from every $T(x_{ni})$ by means of the estimator $\widehat{\theta}_{ni} \stackrel{\text{def}}{=} T(x_{ni})$. We assume that the tabular errors $e_{ni} \stackrel{\text{def}}{=} T(x_{ni}) - f_\theta(x_{ni})$ are independent and have densities k_{ni} defined as follows:

$$
k_{ni}(x) = \begin{cases} \frac{1}{x^2} & |x| \in [i, i+\frac{1}{4}] \\ c_i & |x| \leq \delta_i \\ 0 & \text{elsewhere,} \end{cases} \qquad (2.103)
$$

where c_i and δ_i are such that $\int_{-\infty}^{+\infty} k_{ni}(x)\,dx = 1$ and $\operatorname{Var} e_{ni} = \int_{-\infty}^{+\infty} x^2 k_{ni}(x)\,dx = 1$.[125]

[124] This function arises for instance when we want to estimate the value of the epoch constant c from a table for the equation of time for which the values of the other underlying parameters are known. Cf. Chapter 3, Sections 3.1.3 and 3.2.3.6.

[125] The distribution of e_{ni} is independent of n. Note that $\int_i^{i+\frac{1}{4}} \frac{dx}{x^2} = \frac{1}{4i^2+i} < \frac{1}{2}$ and $\int_i^{i+\frac{1}{4}} x^2 \frac{dx}{x^2} = \frac{1}{4}$, and

We have $\widehat{\theta}_{ni} = f_\theta(x_{ni}) + e_{ni} = \theta + e_{ni}$ and hence $\operatorname{Var}\widehat{\theta}_{ni} = \operatorname{Var} e_{ni} = 1$ for every i. For the weighted estimator $\widehat{\theta} \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{w_{ni}}{W_n}\widehat{\theta}_{ni}$ we take $w_{ni} = \frac{1}{\operatorname{Var} e_{ni}} = 1$ for every i, and $W_n = \sum_{i=1}^{n} w_{ni} = n$. Now define $X_{ni} = (\widehat{\theta}_{ni} - \theta)/\sqrt{n} = e_{ni}/\sqrt{n}$ for every $n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots, n$, and let h_{ni} denote the density of X_{ni}. We have $EX_{ni} = 0$ and $\operatorname{Var} X_{ni} = \frac{1}{n}$ for every n and i, and $\sum_{i=1}^{n} \operatorname{Var} X_{ni} = 1$ for every n. Furthermore, the X_{ni} are asymptotically negligible, since, for given $\epsilon > 0$, $\Pr(|X_{ni}| \geq \epsilon) \leq \operatorname{Var} X_{ni}/\epsilon^2 = 1/n\epsilon^2$ for every n and i. Now take $\epsilon = 1$. Since $\int_{|x|\geq\sqrt{n}} x^2 k_{ni}(x)\,dx$ equals 0 when $i \leq \sqrt{n} - \frac{1}{4}$, $\frac{1}{2}$ when $i \geq \sqrt{n}$, we find that for every n

$$\begin{aligned}\sum_{i=1}^{n} \int_{|y|\geq\epsilon} y^2 h_{ni}(y)\,dy &= \sum_{i=1}^{n} \int_{|y|\geq 1} \sqrt{n} y^2 h_{ni}(\sqrt{n} y)\,dy \\ &= \sum_{i=1}^{n} \int_{|x|\geq\sqrt{n}} \frac{1}{n} x^2 k_{ni}(x)\,dx \\ &\geq \frac{n - \sqrt{n} - 1}{n} \cdot \tfrac{1}{2}. \end{aligned} \qquad (2.104)$$

We conclude that condition (2.100) is not satisfied. Since $\max_{i=1,\ldots,n} \operatorname{Var} X_{ni} = \frac{1}{n} \to 0$ for $n \to \infty$, it follows from the version of the Central Limit Theorem presented above that the distribution of $\sum_{i=1}^{n} X_{ni}$ does *not* converge to the standard normal distribution for $n \to \infty$.

2.A.3 Estimating the Variance of Tabular Errors

Right Ascension. In Section 2.2.2 it was stated that the variance σ^2 of the tabular errors $e_\varepsilon(\lambda) \stackrel{\text{def}}{=} T(\lambda) - \alpha_\varepsilon(\lambda)$ of a right ascension table can be estimated by means of formula (2.28), where n denotes the number of elements of the set of arguments Λ. In this appendix it will be shown that in fact $\frac{1}{n-1} \sum_{\lambda\in\Lambda} (T(\lambda) - \alpha_{\widehat{\varepsilon}}(\lambda))^2$ converges in probability to σ^2 for $n \to \infty$. A series expansion will be given for the right ascension as a function of the solar longitude.[126] Using this expansion, the difference $\alpha_{\widehat{\varepsilon}}(\lambda) - \alpha_\varepsilon(\lambda)$ can for every λ be approximated by a function of the error $e_{\widehat{\varepsilon}} \stackrel{\text{def}}{=} \widehat{\varepsilon} - \varepsilon$ in the estimate of the obliquity. We will then see that $\sum_{\lambda\in\Lambda} (T(\lambda) - \alpha_{\widehat{\varepsilon}}(\lambda))^2 - \sum_{\lambda\in\Lambda} (T(\lambda) - \alpha_\varepsilon(\lambda))^2$ converges in probability to 0 for $n \to \infty$. In a completely analogous way it can be shown that formula (2.42) provides an unbiased estimate of the variance of the errors in a table for the solar equation.

that $\delta_i = \sqrt{\frac{3}{2}\frac{4i^2+i}{4i^2+i-2}}$ and $c_i = \frac{3}{4}/\delta_i^3$. Thus $\lim_{i\to\infty} \delta_i = \sqrt{\frac{3}{2}}$ and $\lim_{i\to\infty} c_i = \frac{1}{6}\sqrt{6}$.

[126] See also North 1976, vol. 3, p. 201.

We write $\alpha = \alpha_\varepsilon(\lambda)$, $t = \tan \frac{1}{2}\varepsilon$, $u = (1-t^2)(e^{2i\lambda}-1)$ and $v = (1+t^2)(e^{2i\lambda}+1)$. Then we have $\cos\varepsilon = \dfrac{1-t^2}{1+t^2}$, and from $\tan\alpha = \cos\varepsilon \cdot \tan\lambda$ it follows that $\dfrac{e^{2i\alpha}-1}{i(e^{2i\alpha}+1)} = \dfrac{u}{iv}$. Equivalently, $(1-\dfrac{u}{v})e^{2i\alpha} = 1+\dfrac{u}{v}$, from which we find

$$e^{2i\alpha} = \frac{v+u}{v-u} = e^{2i\lambda}\frac{(1+t^2e^{-2i\lambda})}{(1+t^2e^{2i\lambda})}. \tag{2.105}$$

By taking logarithms and substituting the Taylor series $\log(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k} x^k$, we obtain

$$\begin{aligned} \alpha_\varepsilon(\lambda) &= \lambda + \frac{180}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}t^{2k}\left(e^{-2ik\lambda}-e^{2ik\lambda}\right) \\ &= \lambda + \frac{180}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}t^{2k}\sin 2k\lambda. \end{aligned} \tag{2.106}$$

Note that the factor $180/\pi$ results from the fact that α and λ are expressed in degrees instead of in radians. Since t^2 is approximately equal to 0.044 the series converges for all λ.

Now let $e_{\widehat{\varepsilon}} = \widehat{\varepsilon} - \varepsilon$ be the error in the estimate $\widehat{\varepsilon}$. For the error $e_t \stackrel{\text{def}}{=} \tan\frac{1}{2}\widehat{\varepsilon} - \tan\frac{1}{2}\varepsilon$ in $t = \tan\frac{1}{2}\varepsilon$ we find by means of a Taylor expansion

$$\begin{aligned} e_t &= \tfrac{1}{2}\frac{\pi}{180}(1+\tan^2\tfrac{1}{2}\varepsilon)e_{\widehat{\varepsilon}} + \tfrac{1}{4}\frac{\pi^2}{180^2}\tan\tfrac{1}{2}\varepsilon(1+\tan^2\tfrac{1}{2}\varepsilon)e_{\widehat{\varepsilon}}^2 + \mathcal{O}(\left(\frac{\pi}{180}e_{\widehat{\varepsilon}}\right)^3) \\ &\approx \frac{\pi}{360}(1+\tan^2\tfrac{1}{2}\varepsilon)e_{\widehat{\varepsilon}}. \end{aligned} \tag{2.107}$$

Note that, since $\tan\frac{1}{2}\varepsilon \approx 0.2$, the second order term is at least a factor 570 smaller than the first order term and can therefore be neglected.

For every λ we can now compute the difference $\alpha_{\widehat{\varepsilon}}(\lambda) - \alpha_\varepsilon(\lambda)$. We have

$$\begin{aligned} \alpha_{\widehat{\varepsilon}}(\lambda) - \alpha_\varepsilon(\lambda) &= \frac{180}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left((t+e_t)^{2k} - t^{2k}\right)\sin 2k\lambda \\ &= \frac{180}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left\{\sum_{j=1}^{2k}\binom{2k}{j}t^{2k-j}e_t^j\right\}\sin 2k\lambda \\ &= -\frac{180}{\pi}2te_t\sin 2\lambda + \frac{180}{\pi}2t^3e_t\sin 4\lambda + \mathcal{O}(t^5) + \mathcal{O}(e_t^2) \quad (t\to 0,\ e_t \to 0) \\ &\approx -t(1+t^2)e_{\widehat{\varepsilon}}\sin 2\lambda. \end{aligned} \tag{2.108}$$

Now we write $A = \sum_{\lambda\in\Lambda}(T(\lambda)-\alpha_\varepsilon(\lambda))^2$, $B = \sum_{\lambda\in\Lambda}(T(\lambda)-\alpha_\varepsilon(\lambda))(\alpha_{\widehat{\varepsilon}}(\lambda)-\alpha_\varepsilon(\lambda))$ and $C = \sum_{\lambda\in\Lambda}(\alpha_{\widehat{\varepsilon}}(\lambda)-\alpha_\varepsilon(\lambda))^2$. Assuming that the distributions of the squares $(T(\lambda)-\alpha_\varepsilon(\lambda))^2$

of the tabular errors satisfy the conditions of the Law of Large Numbers (e.g. if the tabular errors are uniformly bounded or identically distributed),[127] it follows that $\frac{A}{n-1}$ converges in probability to σ^2 for $n \to \infty$. Furthermore, since we can assume that $|\alpha_{\widehat{\varepsilon}}(\lambda) - \alpha_\varepsilon(\lambda)| \leq 1$ for every λ,[128] we find that $\operatorname{Var} B \leq \operatorname{Var} \sum_{\lambda\in\Lambda} (T(\lambda) - \alpha_\varepsilon(\lambda)) = n\sigma^2$. By applying Chebyshev's inequality, it follows that $\frac{B}{n-1}$ converges in probability to 0 for $n \to \infty$. Finally, if Λ is of the form $\{\alpha, 2\alpha, 3\alpha, \ldots, (n-1)\alpha\}$ with $\alpha = 90/n$ and n even, then we have

$$C \approx \sum_{\lambda\in\Lambda} \left(-t(1+t^2)e_{\widehat{\varepsilon}} \sin 2\lambda\right)^2 = t^2(1+t^2)^2 e_{\widehat{\varepsilon}}^2 \cdot \tfrac{1}{2}n. \tag{2.109}$$

Since $e_{\widehat{\varepsilon}}$ has approximately a standard deviation $\widehat{S}_{\widehat{\varepsilon}} = 6.5\sigma/\sqrt{n}$ (see formula 2.27 in Section 2.2.2), it follows that $e_{\widehat{\varepsilon}}$ converges in probability to 0 for $n \to \infty$. Consequently, $\frac{C}{n-1}$ converges in probability to 0 for $n \to \infty$, and

$$\frac{1}{n-1} \sum_{\lambda\in\Lambda} (T(\lambda) - \alpha_{\widehat{\varepsilon}}(\lambda))^2 = \frac{A}{n-1} - \frac{2B}{n-1} + \frac{C}{n-1}$$

converges in probability to σ^2.

Solar Equation. For the solar equation $q_e(\bar{a}) = \arctan\left(\frac{e \sin \bar{a}}{60 + e\cos\bar{a}}\right)$ as a function of the mean solar longitude (see Section 2.2.3, formula 2.42) we arrive in the same way at $q_e(\bar{a}) = \frac{180}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{e}{60}\right)^k \sin k\bar{a}$ for every $\bar{a}$.[129] We then find that $q_{\widehat{e}}(\bar{a}) - q_e(\bar{a}) \approx \frac{3}{\pi} e_{\widehat{e}} \sin\bar{a}$, where $e_{\widehat{e}} \stackrel{\text{def}}{=} \widehat{e} - e$ is the error in the estimate $\widehat{e}$, and that $\sum_{\bar{a}\in\mathcal{A}} (q_{\widehat{e}}(\bar{a}) - q_e(\bar{a}))^2 \approx \frac{9}{\pi^2} e_{\widehat{e}}^2 \cdot \frac{1}{2}n$. As in the case of the right ascension it follows that $\frac{1}{n-1}\sum_{\bar{a}\in\mathcal{A}} (T(\bar{a}) - q_{\widehat{\varepsilon}}(\bar{a}))^2$ converges in probability to σ^2 for $n \to \infty$.

2.A.4 Sums and Integrals

2.A.4.1 $\displaystyle\sum_{k=1}^{n-1} \tan^2\left(\frac{k\pi}{2n}\right) = \tfrac{1}{3}(2n-1)(n-1)$

For every x we have $\cos 4nx = \sum_{k=0}^{2n} a_k \cos^{4n-2k} x$ where $a_k = (-1)^k \frac{4n}{k}\binom{4n-k-1}{k-1} 2^{4n-1-2k}$ for every $0 \leq k \leq 2n$. As $a_{2n} = 1$ we can write

$$\cos 4nx - 1 = \cos^{4n-2} x \cdot \sum_{k=0}^{2n-1} a_k \left(\frac{1}{\cos^2 x}\right)^k. \tag{2.110}$$

[127] See for instance Loève 1977–1978, vol. 1, pp. 286–292.

[128] For values of the obliquity of the ecliptic within the historical range 23;28 to 24, the right ascension varies by less than $7'3''$.

[129] See North 1976, vol. 3, pp. 201–202.

Since $\cos 4nx - 1 = 0$ for $x = \cos\frac{k\pi}{2n}$, $k = 1, 2, 3, \ldots, 4n$, we conclude that the numbers $\dfrac{1}{\cos^2\left(\frac{k\pi}{2n}\right)}$, $k = 1, 2, 3, \ldots, n-1, n+1, n+2, \ldots, 2n$ are the roots of $\sum_{k=0}^{2n-1} a_k Y^k = 0$. Therefore we have

$$S \stackrel{\text{def}}{=} \sum_{\substack{k=1\\k\neq n}}^{2n} \frac{1}{\cos^2\left(\frac{k\pi}{2n}\right)} = -\frac{a_{2n-2}}{a_{2n-1}} = \tfrac{1}{3}(2n+1)(2n-1). \tag{2.111}$$

It follows that

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2\left(\frac{k\pi}{2n}\right)} = \sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{2n}\right)} = \tfrac{1}{2}(S-1) = \tfrac{2}{3}(n^2-1) \tag{2.112}$$

and that

$$\sum_{k=1}^{n-1} \tan^2\left(\frac{k\pi}{2n}\right) = \left(\sum_{k=1}^{n-1} \frac{1}{\cos^2\left(\frac{k\pi}{2n}\right)}\right) - (n-1) = \tfrac{1}{3}(2n-1)(n-1). \tag{2.113}$$

More information about properties of trigonometric series can for instance be found in the textbook by Bromwich.[130]

2.A.4.2 Various Integrals

The following two elementary integrals for $c > 0, c \neq 1$ can be calculated by means of the substitution $x = \tan\lambda$.

$$\int_0^{\frac{1}{2}\pi} \frac{\tan^2\lambda\, d\lambda}{1 + c^2\tan^2\lambda} = \frac{\pi}{2c(c+1)} \tag{2.114}$$

$$\int_0^{\frac{1}{2}\pi} \frac{\tan^2\lambda\, d\lambda}{(1 + c^2\tan^2\lambda)^2} = \frac{\pi}{4c(c+1)^2} \tag{2.115}$$

The following elementary integrals for $c > 1$ can be calculated by means of the substitution $y = \tan\frac{1}{2}x$.

$$\int_0^{\pi} \frac{\sin^2 x\, dx}{(c^2 + 2c\cos x + 1)^2} = \frac{\pi}{2c^2(c^2-1)} \tag{2.116}$$

$$\int_0^{\pi} \frac{(c + \cos x)\, dx}{c^2 + 2c\cos x + 1} = \frac{\pi}{c} \tag{2.117}$$

[130] See Bromwich 1926, pp. 202–213.

Further Research

In this chapter four estimators for unknown parameter values in astronomical tables have been discussed. Further research needs to be done on various aspects of these estimators.

1. We would like to introduce a type of asymptotics that takes into account that the variance of the tabular errors will usually be smaller when the total number of tabular values is larger. When this type of asymptotics is used, the weighted estimator will generally be asymptotically unbiased.
2. We would like to compute a Fourier estimator for the translation parameter of a planetary equation table which makes use of the information contained in estimated Fourier coefficients $\widehat{a}_k$ and $\widehat{b}_k$ for every value of k. (The Fourier estimator presented in Section 2.3 makes use only of the estimated coefficients $\widehat{a}_1$ and $\widehat{b}_1$.) Since the estimators for the Fourier coefficients described in Section 2.3 are independent, we can compute a weighted average of the estimators $\widehat{\lambda_A}(k) \stackrel{\text{def}}{=} -\arctan(\widehat{a}_k/\widehat{b}_k)/k$ for the translation parameter λ_A in order to obtain a more accurate estimator. Generally the errors in the estimated Fourier coefficients overwhelm the Fourier coefficients themselves as k increases. Therefore we will need a criterion for deciding for which values of k the estimator $\widehat{\lambda_A}(k)$ should be included in the weighted average.
3. We would like to estimate the variance of the tabular errors in a particular table from estimates for the Fourier coefficients. The properties of the estimators for the variance indicated in Section 2.3.3 should be investigated. Furthermore, it may be possible to find a better estimator for the variance of the tabular errors.
4. We would like to develop a statistical theory based on the Least Number of Errors Criterion. As was indicated in Section 2.5, it may be possible to develop such a theory if assumptions are made concerning the distribution of the tabular errors. It would also be interesting to investigate the results of the Least Number of Errors Criterion if, in situations where the number of errors in the table is relatively large, δ as defined in Section 2.5 is taken larger than the unit of the tabular values. Finally, it would be useful to investigate how estimates of multiple unknown parameters could be determined according to the Least Number of Errors Criterion.

Further research also needs to be done on the distribution and dependency of tabular errors; see the discussion in Section 1.2.4.

Chapter 3

Case Study: the Equation of Time

3.1 Introduction

In this chapter I will discuss four tables from Greek and Islamic sources for the so-called "equation of time", a complicated function based on four astronomical parameters, which was tabulated in different ways. Up till now tables for the equation of time have only been recomputed successfully in exceptional cases. By means of the statistical estimators discussed in the previous Chapter (in particular the least squares estimator) and various "ad hoc methods" I will demonstrate that it is possible to recover the mathematical structure and the underlying parameters of a table for the equation of time. First it will be explained how the equation of time can be calculated within the framework of Ptolemy's planetary theory. Then important publications by modern scholars concerning the equation of time in mediaeval sources will be surveyed and methods for the analysis of tables for the equation of time will be given. Finally the tables for the equation of time in Ptolemy's *Handy Tables*, in the Greek papyrus London 1278, in Kushyār ibn Labbān's *Jāmiʿ Zīj* and in the *Baghdādī Zīj* will be analysed extensively.

3.1.1 Theoretical Exposition

Astronomers from Antiquity and the Middle Ages knew that true solar time (as can be read from a sundial) and astronomical or mean solar time (which was used to determine the positions of the planets from tables) are not generally equal. Although the difference is about half an hour at most, it has to be taken into account when an accurate position of a fast celestial body like the moon is needed, e.g. in the calculation of the time of an eclipse. To convert true solar time to mean solar time or vice versa one has to add or subtract a quantity which varies throughout the year. Many Greek and Byzantine astronomers called this quantity ἡ παρὰ τὴν ἀνισότητα τῶν νυχθημέρων διαφορά (the difference due to the inequality of the days and nights),[1] but Ptolemy did not use any particular designation. The Arabic name was تَعْدِيلُ ٱلْأَيَّامِ بِلَيَالِيهَا (correction of the days and

[1]See for example the *Great Commentary on Ptolemy's Handy Tables* by Theon of Alexandria: Mogenet & Tihon 1985, p. 98, lines 10–11 and p. 119, lines 10–11.

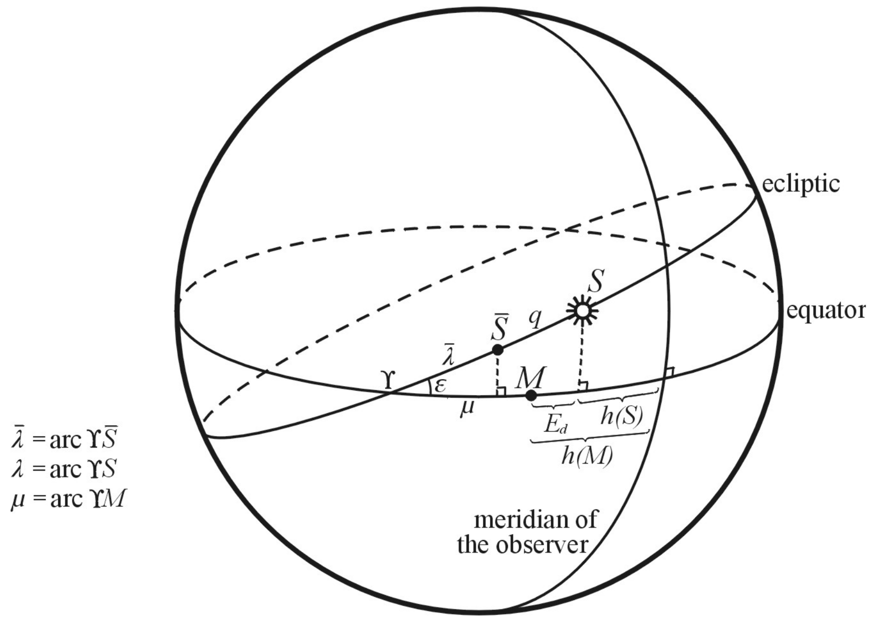

Figure 3.1: Geometrical explanation of the equation of time

their nights), which was translated into Latin as *equatio dierum cum noctibus suis.* In modern astronomy the correction is called the *equation of time.* Ptolemy's *Handy Tables* and many mediaeval Islamic astronomical handbooks contain a table for the equation of time as a function of the solar position.

To understand the concept of the equation of time and the way in which the equation can be computed, we will have to consider both annual and daily effects.[2] The tropical year is defined by Ptolemy as the period of time in which the true sun S, which moves on the ecliptic at a variable apparent speed, returns to a particular equinox or solstice.[3] The so-called "ecliptical mean sun" $\bar{S}$, which moves on the ecliptic, and the "equatorial mean sun" M, which moves on the equator, both have a uniform motion and a period of precisely one tropical year.[4] $\bar{S}$ and M will be well defined as soon as we fix their position at a certain point in time with respect to the position of the true sun S. Let λ denote the ecliptical longitude of the true sun, $\bar{\lambda}$ the ecliptical longitude of the ecliptical mean sun, and μ the right ascension of the equatorial mean sun. λ, $\bar{\lambda}$, and μ are all measured from

[2] I will explain the equation of time as far as possible in the terminology of Ptolemy's solar model, which was in use throughout the Middle Ages. Explanations of the equation of time as used by Ptolemy can also be found in Neugebauer 1975, vol. 1, pp. 61–68, and Pedersen 1974, pp. 154–158. For a modern treatment of the equation of time, see Smart 1977, pp. 146–150.

[3] Heiberg 1898–1903, vol. 1, pp. 192–193; Toomer 1984, p. 132. Note that Ptolemy considers both the length of the tropical year and the tropical longitude of the solar apogee to be constants.

[4] The mean suns cannot be found explicitly in Ptolemy's work. They are modern concepts, which I use for the sake of simplicity. Instead of the ecliptical mean sun Ptolemy consistently used the equivalent concept "position of the sun in mean motion". Instead of the equatorial mean sun he applied the concept of simultaneously rising arcs of the equator and the ecliptic. As we will see below, the ecliptical and equatorial mean suns need not pass the equinoxes simultaneously.

the vernal point. Note that, since $\bar{S}$ and M have a uniform motion and the same period, $\bar{\lambda}$ and μ increase linearly at the same rate. Therefore there is a constant c such that $\bar{\lambda} + c = \mu$ at any moment. λ and $\bar{\lambda}$ are related through $\bar{\lambda} = \lambda + q$, where q is a variable quantity called the "solar equation". A more extensive discussion of c and formulae for q will be given below.

Now we will consider the daily effects. Time is usually reckoned from midday, i.e. from the upper culmination of the sun. Thus true solar time is defined as the hour angle $h(S)$ of the true sun, mean solar time is defined as the hour angle $h(M)$ of the equatorial mean sun.[5] The equation of time E_d, where the subscript d indicates that the equation is measured in degrees, can now be defined as the difference $h(S) - h(M)$ between true and mean solar time (see Figure 3.1). E_d is not a constant for two reasons. Firstly, the true sun has a variable speed, whereas the equatorial mean sun moves uniformly. Secondly, the true sun moves on the ecliptic, the equatorial mean sun on the equator, and in general equal arcs of the equator and the ecliptic do not pass the meridian in equal time spans.

From $E_d = h(S) - h(M)$ it follows that at any moment E_d equals the difference in the right ascension of the equatorial mean sun and the true sun:[6]

$$E_d = \mu - \alpha(\lambda) = \bar{\lambda} - \alpha(\lambda) + c. \tag{3.1}$$

Here $\alpha(\lambda)$ denotes the right ascension of λ, i.e. the arc (measured in eastward direction) between the vernal point and the orthogonal projection of the point on the ecliptic that has longitude λ onto the equator. α has a single underlying parameter, namely the obliquity of the ecliptic ε. For $\lambda \in [0, 90\rangle$ the right ascension is given by the modern formula

$$\alpha(\lambda) = \arctan\left(\cos\varepsilon \cdot \tan\lambda\right). \tag{3.2}$$

For $\lambda \in [90, 360]$ the right ascension can be determined using the symmetry relations $\alpha(180 - \lambda) = 180 - \alpha(\lambda)$ and $\alpha(180 + \lambda) = 180 + \alpha(\lambda)$.

As was mentioned before, we have $\bar{\lambda} = \lambda + q$, where the solar equation q depends on the solar apogee λ_A and the solar eccentricity e. As a function of λ, q can be expressed as[7]

$$q(\lambda) = \arcsin\left(\frac{e}{60}\sin(\lambda - \lambda_A)\right). \tag{3.3}$$

When the solar equation is expressed as a function of $\bar{\lambda}$, I will use the symbol $\bar{q}$. We have

$$\bar{q}(\bar{\lambda}) = \arctan\left(\frac{e \cdot \sin(\bar{\lambda} - \lambda_A)}{60 + e \cdot \cos(\bar{\lambda} - \lambda_A)}\right). \tag{3.4}$$

Note that $q = \bar{q} = 0$ (and hence $\lambda = \bar{\lambda}$) whenever $\lambda = \lambda_A$ or $\lambda = \lambda_A + 180°$.

[5]The hour angle of a heavenly body X is the spherical angle between the meridian of the observer and the meridian through X, measured in westward direction. In this section it will always be measured in equatorial degrees (as opposed to hours).

[6]Note that the right ascension is measured in the opposite direction of the hour angle. Since the equatorial mean sun is positioned on the equator, its right ascension is equal to its equatorial longitude μ.

[7]See for instance Pedersen 1974, pp. 144–154 for an extensive description of Ptolemy's solar model.

The constant c occurring in formula (3.1) determines the synchronization of the ecliptical mean sun and the equatorial mean sun. In modern astronomy c is usually taken to be zero; this implies that $\bar{S}$ and M pass the vernal equinox simultaneously.[8] Ptolemy and mediaeval astronomers used several other methods to fix c, some of which involved convenient choices of the so-called "epoch", the starting point of the mean motion tables.[9] For example, Ptolemy implicitly assumed that mean and true solar time at epoch were equal.[10] At the epoch of his *Handy Tables* the equation of time happened to be approximately at its yearly minimum. By choosing c appropriately, this minimum could be made equal to $0;0,0^{\mathrm{h}}$, whence all equation of time values became non-negative.[11] In the sequel c will be called "epoch constant".

In Ptolemy's *Handy Tables* and in most mediaeval astronomical handbooks which contain a table for the equation of time, the tabulated quantity is E_h, the equation of time expressed in hours, rather than E_d.[12] To convert E_d to E_h, E_d was usually divided by 15, in agreement with the identity $24^{\mathrm{h}} = 360°$. However, since the sun's daily motion in the ecliptic amounts to approximately 0;59,8°/day, a more accurate conversion factor is $\frac{360;0+0;59,8}{24} \approx 15;2,28°/\mathrm{hour}$. This number was used by al-Kāshī in his Khāqānī Zīj.[13] The maximum error introduced by taking the factor 15 instead of 15;2,28 is only 5 seconds of time. I will write $E_h = \frac{1}{D} E_d$, where D ordinarily can only take the values 15 and 15;2,28.

Thus we find the following formula for the equation of time expressed as a function of the true solar longitude λ:

$$E_h(\lambda) = \frac{1}{D}\left(\lambda + q(\lambda) - \alpha(\lambda) + c\right). \tag{3.5}$$

When the equation of time is expressed as a function of the mean solar longitude $\bar{\lambda}$, I will use the symbol $\bar{E}_h$. We have

$$\bar{E}_h(\bar{\lambda}) = \frac{1}{D}\left(\bar{\lambda} - \alpha\left(\bar{\lambda} - \bar{q}(\bar{\lambda})\right) + c\right). \tag{3.6}$$

The general shape of the equation of time can be seen from Figure 3.2, which plots a recomputation of al-Kāshī's table for the equation of time as found in the Khāqānī Zīj.[14]

[8]Note, however, that at the moment when $\bar{S}$ and M pass the vernal equinox, the true sun has an ecliptical position $\lambda = -\bar{q}(0)$ which is generally not equal to zero. If we apply the equation of time to convert true solar time at particular European locations to M.E.T. (Middle European Time), different values of c would be needed for, for example, Amsterdam and Berlin.

[9]Five different methods for fixing c are mentioned by Ibn Yūnus in his Ḥākimī Zīj; see the manuscript Leiden Ms. 1057, p. 91.

[10]In fact, Ptolemy defined the equation of time to be the difference between the true solar time and the mean solar time that had passed since epoch. See the references given in footnote 2.

[11]Cf. Neugebauer 1975, vol. 1, pp. 63–65 and vol. 2, pp. 984–985. If the equation of time at epoch is not close to its minimum or maximum value, the mean solar longitude at epoch may be adjusted in order to obtain an always positive or an always negative equation of time; see Neugebauer 1962, pp. 63–65.

[12]One of the very few exceptions to this rule is al-Battānī's table for the equation of time in the Ṣābiʾ Zīj; see Nallino 1899–1907, vol. 2, pp. 61–64.

[13]See Kennedy 1988, p. 5. My impression is that practically all earlier Islamic astronomers used the conversion factor 15.

[14]Cf. Kennedy 1988.

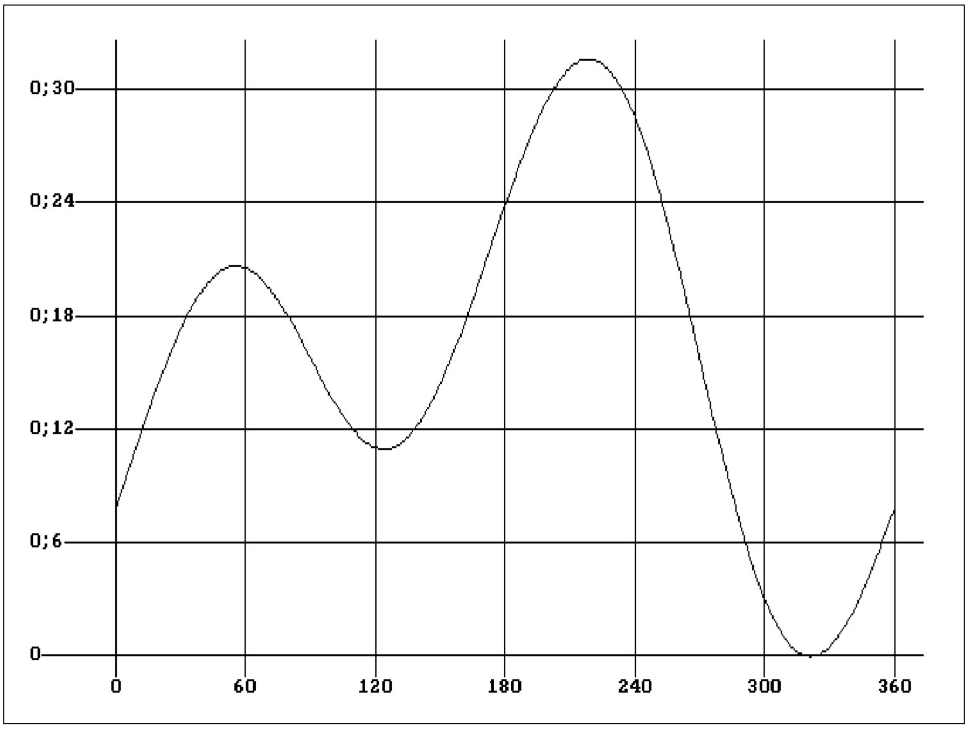

Figure 3.2: Recomputation of al-Kāshī's equation of time

The independent variable is the true solar longitude λ, the underlying parameter values are $\varepsilon = 23;30$, $e = 2;6,9$, $\lambda_A = 90;0$, $c = 3;57,34$, the conversion factor $D = 15;2,28$. The equation of time as a function of the mean solar longitude has the same general shape. In fact, in a graphical representation like Figure 3.2, it would not be possible to distinguish between two tables for the equation of time based on the same parameter values but having different independent variables.

Ptolemy's *Handy Tables* and many mediaeval Islamic astronomical handbooks contain a table for the equation of time, usually in the same section as the tables for the mean solar motion and the solar equation. Both the true solar longitude and the mean solar longitude occur as the independent variable of tables for the equation of time.[15] It can be noted that, as far as the computation of a table for the equation of time or its application is concerned, it makes little difference whether the independent variable is the true or the mean solar longitude. To calculate the equation of time as a function of the true solar longitude, one need only perform a simple addition and / or subtraction of right ascension and solar equation values. The right ascension values can readily be taken from a sufficiently accurate table for $\alpha(\lambda)$. If a table for $q(\lambda)$ is available, the solar equation values can readily be taken from it except if the solar apogee does not occur in the argument column (in that case interpolation is needed). If only a table for $\bar{q}(\bar{\lambda})$ is available, inverse interpolation in this table or a separate calculation of $q(\lambda)$ is necessary.

[15]In the remainder of this chapter, we will see that the tables for the equation of time in the *Handy Tables* and in the Baghdādī Zīj have the true solar longitude as their independent variable, whereas the table in Kushyār's Jāmiʿ Zīj has the mean solar longitude as its indepedendent variable.

Note that the solar equation tables in most zījes display $\bar{q}(\bar{\lambda})$ instead of $q(\lambda)$.

To calculate the equation of time as a function of the mean solar longitude, one always needs to perform interpolation in a right ascension table. If a table for $\bar{q}(\bar{\lambda})$ is available, the solar equation can readily be taken from it except if the solar apogee does not occur in the argument column (in that case interpolation is needed). If only a table for $q(\lambda)$ is available (we have seen above that this is an unlikely situation), inverse interpolation or a separate calculation of $\bar{q}(\bar{\lambda})$ will be necessary.

The equation of time can be applied as follows:

- Assume that on a certain date the true solar time $h(S)$ is known.[16]
- Look up the mean solar position $\bar{\lambda}$ for the time $h(S)$ in the tables for mean solar motion.[17]
- Now, if the argument of the table for the equation of time is the mean solar longitude, find $\bar{E}_h(\bar{\lambda})$ in the table and calculate the mean solar time $h(M)$ from $h(M) = h(S) - \bar{E}_h(\bar{\lambda})$. Otherwise, calculate the true solar position λ for the time $h(S)$ by means of the table for the solar equation and the formula $\lambda = \bar{\lambda} - \bar{q}(\bar{\lambda})$. Next, find $E_h(\lambda)$ in the table for the equation of time and calculate the mean solar time $h(M)$ from $h(M) = h(S) - E_h(\lambda)$.
- Adjust the mean solar position $\bar{\lambda}$, which we initially determined for the true solar time $h(S)$, by adding the mean solar motion during the time span $h(M) - h(S)$.[18]

3.1.2 State of the Art

Ptolemy's *Handy Tables* and numerous extant astronomical handbooks from Islamic, Byzantine and Western European origin contain a table for the equation of time. Most of these tables are essentially different, i.e. they are based on different values for the four underlying parameters or they have been computed according to different algorithms. Extensive use of interpolation and the application of approximate methods cannot be ruled out in advance.

Occasionally, information about the computation and/or underlying parameter values of equation of time tables can be found in primary sources. For instance, Theon's *Great Commentary on the Handy Tables* explains the computation of the equation of time table

[16] A mediaeval astronomer determined $h(S)$ from an observation of the altitude of the sun or a bright star. Large sets of tables for timekeeping were available, by means of which for instance the number of hours since sunrise could be determined as a function of the solar altitude and longitude (cf. King 1973). To use these tables, one first had to determine a rough value of the true solar position for the date concerned. In the last step below a more accurate value will be obtained.

[17] Note that we introduce a small error here, since the argument of the mean motion tables is mean solar time. We will correct this error after determining the equation of time. Cf. al-Baghdādī's explanatory text in Section 3.4.1.

[18] Cf. footnote 17. From a modern point of view, we would iterate the entire process in order to obtain an arbitrarily good approximation to $h(M)$. For mediaeval purposes the one-step algorithm as described was accurate enough.

in the *Handy Tables*.[19] In his *Ḥākimī Zīj*, Ibn Yūnus describes the computation of the equation of time extensively and gives the methods applied by various important Islamic astronomers.[20] Kushyār ibn Labbān describes the method he used to compute his table for the equation of time in the *Jāmiʿ Zīj*.[21] However, the information presented is usually insufficient for determining all underlying parameter values of the table for the equation of time concerned and for establishing the way in which the table was computed.

Very few studies about the equation of time in ancient and mediaeval sources have been published, and hardly any tables have been properly analysed. Nevertheless, the first serious attempt to determine the underlying parameter values of a table for the equation of time was made as early as 1799. According to a letter from Burckhardt to Zach, Gauss applied his newly invented method of least squares to the table for the equation of time by Ulugh Beg.[22] His objective was purely mathematical and he did not draw historical conclusions. As far as I know, it was not until recently that the method of least squares was again used to determine unknown parameter values in astronomical tables.

The following authors who published about the equation of time in Greek and Islamic sources should be mentioned:
Rome's *Le problème de l'équation du temps chez Ptolémée* gives an excellent overview of the information in primary sources concerning the equation of time as used by Ptolemy.[23] Neugebauer summarizes the information found in the Almagest and makes some important remarks about the table for the equation of time in the Handy Tables.[24] North computes series expansions for the equation of time as a function of the true and mean solar longitude. He uses the expressions for the coefficients to determine approximations for the parameters.[25] Kennedy analysed two Islamic tables for the equation of time in a recent article.[26]

From the above it appears that there is no historical overview of the computational aspects (algorithms, parameter values, relation with extant right ascension and solar equation tables, use of interpolation) of tables for the equation of time. Likewise very little is known about the relation between tables for the equation of time in different zījes.

In the remainder of this chapter I will outline a number of methods for analysing tables for the equation of time (Section 3.1.3) and I will give various examples of applications

[19] Mogenet & Tihon 1985, pp. 98–100 and 163–164. In Section 3.2 we will see that Theon's explanation is not in agreement with the actual method of computation.

[20] Leiden Ms. 1057, pp. 86–92.

[21] Istanbul Fatih 3418, ff. 7^r–7^v or Yahya 1986, p. 122.

[22] See *Allgemeine geographische Ephemeriden* **3** (1799), pp. 179–185, and C.F. Gauss, *Werke*, vol. 12, pp. 64–68. Ulugh Beg's table is extant in Bodleian Ms. Pocock. 226, f. 94^r and many other manuscripts of the Zīj i-Sulṭānī.

[23] Rome 1939.

[24] Neugebauer 1975, vol. 1, pp. 61–68 and vol. 3, 984–986. See also Neugebauer 1958, pp. 109–111.

[25] North 1976, vol. 3, pp. 201–205. North indicates that the results of the estimations are not very accurate. Furthermore, it turns out to be difficult to distinguish between true and mean solar longitude as the independent variable of a given table for the equation of time. Mercier also gives a series expansion of the equation of time, but does not use it for approximating the parameters (Mercier 1985, p. 24).

[26] Kennedy 1988. Kennedy recomputed al-Kāshī's table for the equation of time successfully. For Kushyār's table I will suggest a different method of computation in Section 3.3.

of these methods (Sections 3.2 to 3.4). From the results preliminary conclusions will be drawn about the relations between tables for the equation of time in different zījes (Section 3.5).

3.1.3 Analysis of tables for the equation of time

From the tabular values of a given table for the equation of time we want to extract information concerning the method by which the table was computed. In particular, we want to know:

- the independent variable (mean or true solar longitude);
- the values of the four underlying parameters: the obliquity of the ecliptic ε, the solar apogee λ_A, the solar eccentricity e, and the epoch constant c;
- the conversion factor D (15 or 15;2,28; in most cases we expect $D = 15$);
- the accuracy of the underlying tables for the solar equation and the right ascension.

Most of this information cannot be obtained directly. For instance, the differences between tables that are based on the same set of parameters but have different independent variables or different conversion factors are not perceptible in a graphical representation like Figure 3.2. Furthermore there is no easy way to obtain an accurate approximation for the values of the parameters; only in exceptional cases will trial-and-error lead to good results.

To obtain the desired information we may use the following properties and methods:

1. Many tables for the equation of time appear to have been calculated using some type of interpolation (mostly linear), possibly in an intermediate stage of the calculation. Interpolation can be recognized by studying the first differences of the table. In typical examples every 5th or 6th tabular value was calculated exactly, the intermediate values were calculated by means of interpolation. Tabular values calculated exactly (i.e. without the use of interpolation) will be called "*nodes*", other values, "*internodal values*". In the analysis of a table the internodal values can usually be disregarded. However, sometimes they can be used to correct scribal errors in the nodes.
2. If the final sexagesimal digit of (nearly) all tabular values or all nodes is a multiple of 4, we can safely conclude that for the table under consideration $D = 15$ and that the corresponding values for E_d were given to one sexagesimal place less, e.g. $E_h = 0;18,20$ derives from $E_d = 4°35'$.[27] Note that if only the final sexagesimal digit of the nodes is a multiple of 4, interpolation must have been performed after the division by 15.
3. The symmetry relations $\alpha(180 - \lambda) = 180 - \alpha(\lambda)$ and $\alpha(180 + \lambda) = 180 + \alpha(\lambda)$ for the right ascension, and $q(\lambda) = q(2\lambda_A - \lambda)$, $\bar{q}(\bar{\lambda}) = \bar{q}(2\lambda_A - \bar{\lambda})$, $q(\lambda) = q(2\lambda_A + 180 - \lambda)$, and $q(\lambda) = -q(180 + \lambda)$ for the solar equation, lead to the following properties of the

[27] In an accurately computed trigonometric table the final digits are very close to uniformly distributed (see Section 1.2.4). This implies, for example, that the probability of more than 55 multiples of 4 among 60 final digits equals 10^{-29} approximately.

equation of time E_h and $\bar{E}_h$ (in (3.8), n denotes half the number of available equation of time values):

$$E_h(\lambda_A) = E_h(\lambda_A + 180) \text{ and } \bar{E}_h(\lambda_A) = \bar{E}_h(\lambda_A + 180) \tag{3.7}$$

$$\sum_{i=1}^{2n} E_h(\frac{180i}{n}) = \frac{2nc}{D} \quad \text{for every } n, \quad \text{and} \int_0^{2\pi} \bar{E}_h(\bar{\lambda})\, d\bar{\lambda} = \frac{2\pi c}{D} \tag{3.8}$$

$$E_h(\lambda) + E_h(180 - \lambda) + E_h(\lambda + 180) + E_h(360 - \lambda) = \frac{4c}{D} \qquad \text{for every } \lambda \tag{3.9}$$

$$E_h(\lambda) + E_h(\lambda + 180) = \frac{2}{D}\,(\lambda - \alpha(\lambda) + c) \qquad \text{for every } \lambda \tag{3.10}$$

$$E_h(\lambda) - E_h(\lambda + 180) = \frac{2}{D} q(\lambda) \qquad \text{for every } \lambda. \tag{3.11}$$

Using (3.7) and, if necessary, interpolation between tabular values, a rough estimate of λ_A can be obtained.[28] Using (3.8) and (3.9) accurate estimates for c can be calculated.[29] Using (3.10) and (3.11) the underlying right ascension and solar equation can be extracted from a table for the equation of time as a function of the true solar longitude (see for instance Sections 3.2.3.4 and 3.2.3.5).

4. Using a least squares estimation as explained in Section 2.4, accurate estimates of ε, λ_A, e, and c can be obtained. Note that the formula according to which the table was computed must be known before the parameters can be estimated. Since this is not always the case, it may be necessary to perform the estimation for various possible formulae, for instance for all four possible combinations of the independent variable (λ or $\bar{\lambda}$) and the conversion factor D (15 or 15;2,28). The plausibility of a certain hypothesis concerning the method of computation can be judged from the minimum obtainable standard deviation of the residuals: for tabular values that are given to k sexagesimal fractional digits this standard deviation should not be much larger than $17 \cdot 60^{-(k+1)}$.[30] Using this property a least squares estimation distinguishes very clearly between the true and mean solar longitude as the independent variable. A decision about the conversion factor can be made from the historical plausibility of the estimates of the parameters. Note that a least squares estimation can only be applied if certain conditions concerning the tabular errors (namely that they are independent and have mean zero and identical standard deviations) are satisfied. I will not always mention this explicitly.

In the following Sections I will carry out extensive analyses of the tables for the equation of time in Ptolemy's *Handy Tables*, in Kushyār ibn Labbān's *Jāmiʿ Zīj* and in the *Baghdādī Zīj*. Thereby I will demonstrate that, by means of the four methods indicated above, the underlying parameter values and the precise method of computation used for each of the three tables can be determined. Only in the case of Ptolemy will I also discuss the textual information found in primary sources in detail.

[28] For an example see Kennedy 1988, pp. 3–4.

[29] The accuracy of these estimates can easily be calculated. I have not included the calculations, since they are not of particular interest.

[30] This is the standard deviation of a uniform distribution on an interval of length 60^{-k}.

3.2 Ptolemy

In this section I analyse the table for the equation of time from Ptolemy's *Handy Tables*, extant in a large number of Byzantine manuscripts. Section 3.2.1 sketches the available historical information concerning the equation of time in antiquity in general. Section 3.2.2 gives a description of the table to be analysed, lists the specific historical information with regard to its mathematical structure and underlying parameters, and includes some information regarding the method of rounding that Ptolemy applied. The technical analysis of the table follows in Section 3.2.3 and my conclusions are summarized in Section 3.2.4. In Section 3.2.5 I will examine the table for the equation of time in the papyrus London 1278, which was studied by Neugebauer and which, as I will show, is related to the table in the Handy Tables.

3.2.1 Historical context and sources

Ptolemy is the most important astronomer of antiquity.[31] He lived in the second century A.D. in Alexandria, where he made a large number of observations which are recorded in his works. Ptolemy expanded Hipparchus' solar and lunar models and was the first to develop satisfactory models for the other planets. He compiled sets of tables for the easy determination of planetary positions at arbitrary points in time. Ptolemy's most important astronomical works, the more theoretical *Almagest* (originally entitled Μαθηματικὴ Σύνταξις) and the practical *Handy Tables*, were in use up to the end of the Middle Ages.

In the second half of the fourth century Theon of Alexandria (father of the female mathematician Hypatia) wrote various commentaries on Ptolemy's work. Theon was involved in higher education, possibly at the Alexandria Museum. His commentaries were written to comply with the needs of his students, some of whom "were not able to follow a multiplication or division of [sexagesimal] numbers."[32] Little is known about original scientific achievements of Theon.

In most extant manuscripts we find Ptolemy's *Handy Tables* preceded by Theon's *Small Commentary* on these tables.[33] However, since Ptolemy's own introduction, which has only come down to us separately, is in good agreement with what we find in the tables, we can conclude that most of the *Handy Tables* were in fact written by Ptolemy himself and not by Theon.[34] In his *Great Commentary* Theon describes extensively how the tables were computed and how they were derived from the tables in the *Almagest*. In many cases his description is correct, but we will see that Theon did not know the exact way in which the table for the equation of time had been computed.

[31]Most of the following information was taken from the *DSB*-articles "Ptolemy" and "Theon of Alexandria" by G.J. Toomer.

[32]Tihon 1978, p. 199, lines 5–6 (Greek) and p. 301, lines 7–9 (French translation).

[33]See below for more information about Theon's *Small* and *Great Commentary*.

[34]Ptolemy's *Introduction to the Handy Tables* is extant in at least 14 manuscripts which are listed in Heiberg 1907, pp. clxxv–clxxix, and is edited in *idem,* pp. 159–185. The edition and French translation in Halma 1822–1825, vol. 1, pp. 1–26 is unreliable.

Historical information with regard to the equation of time as used by Ptolemy is found in the following primary sources: the *Almagest*, the *Handy Tables*, Ptolemy's own *Introduction to the Handy Tables*, and Theon's *Commentary on the Almagest*, *Great Commentary on the Handy Tables*, and *Small Commentary on the Handy Tables*.[35] Much of this information was summarized by A. Rome in his article *Le problème de l'équation du temps chez Ptolémée*.[36] I now give short descriptions of the sections from the various primary sources which contain information about the equation of time. I will refer to them in Section 3.2.2.

In Section 9 of Book III of the *Almagest* Ptolemy examines the problem of the "inequality of the days".[37] He explains the reasons for this inequality and determines the maximum deviation. Furthermore he shows how a given time span in true solar days can be converted to mean solar days.[38] Values for the mean and true solar position at the beginning of the Egyptian year 1 Nabonassar are presented. No mention is made of a table for the equation of time.

Theon of Alexandria's *Commentary on the Almagest* follows the Almagest closely and does not give important new information. An example is presented which utilizes the initial values mentioned in the *Almagest*.[39]

The set of astronomical tables known as the *Handy Tables* (Πρόχειροι Κανόνες) is extant in a large number of Greek manuscripts. Three manuscripts of the Bibliothèque Nationale in Paris were used by Halma in his edition and French translation published in 1822–1825. Stahlman consulted the manuscript Vatican gr. 1291 for his English edition of the tables (Stahlman 1959). Neugebauer gave a comprehensive technical description of the *Handy Tables* in *A History of Ancient Mathematical Astronomy*.[40] Recently Tihon published an extensive description of the oldest manuscripts of the *Handy Tables* and of the tables occurring in these manuscripts.[41] I myself consulted Leiden BPG 78, one of the oldest extant manuscripts. As was indicated above, most of the contents of the *Handy Tables* probably originates from Ptolemy, although some of the tables are obviously later (Byzantine) additions. We will also have to investigate the tables for the right ascension and the solar equation in the *Handy Tables*, since they may be related to the table for the equation of time.

In his own *Introduction to the Handy Tables*[42] Ptolemy first gives a partial table of contents, which includes the table for the equation of time. He mentions the Era Philip as the epoch of the mean motion tables and $65°30'$ as the solar apogee. Next he explains

[35] Note that in works written before Ptolemy we find no reference to the equation of time as used by Ptolemy (cf. Neugebauer 1975, vol. 2, pp. 584 and 766).

[36] Rome 1939.

[37] Heiberg 1898–1903, vol. 1, pp. 258–263; Toomer 1984, pp. 169–172.

[38] The procedure given by Ptolemy is an approximation; cf. Rome 1939, p. 216; and Pedersen 1974, p. 157.

[39] Since Theon's *Commentary on the Almagest* is only available in a Greek edition, I have not been able to study it in detail. The commentary concerning the "inequality of the days" is found in Rome 1931–1943, vol. 3, pp. 917–942.

[40] Neugebauer 1975, vol. 2, pp. 969–1028.

[41] Tihon 1992.

[42] Heiberg 1907, pp. 159–185; Halma 1822–1825, vol. 1, pp. 1–26.

how to calculate the true solar position and how to convert true solar time to mean solar time using the table for the equation of time.[43]

In the *Great Commentary on the Handy Tables*[44] Theon of Alexandria discusses the layout, computation and use of the *Handy Tables* extensively. In Chapter 2 of Book 1 he describes the layout of the table for the right ascension and the equation of time and the type of interpolation which was used to derive the right ascension values from those in the *Almagest*.[45] Both descriptions are in full agreement with what we find in the extant manuscripts of the *Handy Tables*. Next he describes how the equation of time could be computed from the true solar position using the right ascension table and inverse interpolation in the solar equation table, and following Ptolemy's method in the *Almagest*. Again the solar apogee is taken to be $65°30'$.[46]

In Chapter 9 of the *Great Commentary* Theon explains the conversion from local true solar time given in seasonal hours to Alexandria mean solar time.[47] This conversion consists of three steps:

1. conversion from seasonal to equinoctial hours using the table for the length of the seasonal hours for the appropriate climate.
2. conversion from local to Alexandria true solar time by adding or subtracting the difference in geographical longitude divided by 15.
3. conversion from true to mean solar time by adding the value of the equation of time corresponding to the previously found true solar position.

In Chapter 5 of the *Small Commentary on the Handy Tables*,[48] which Theon wrote after the *Great Commentary*, the above-mentioned conversion from true to mean solar time is illustrated with a numerical example.[49] For a particular date in the year 360 A.D. Theon arrives at a true solar position of 175° and mentions that the equation of time amounts to approximately 7 minutes. (The value occurring in the table for the equation of time in the *Handy Tables* is $6^{m}55^{s}$.)

I will refer to the above-mentioned sources in the section that presents specific historical information about the mathematical structure and underlying parameters of Ptolemy's table for the equation of time (3.2.2.2 below).

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[43] Heiberg 1907, pp. 162–163; Halma 1822–1825, vol. 1, p. 5.

[44] Book I of the *Great Commentary* is edited, translated and commented upon in Mogenet & Tihon 1985; Book II and III, in Tihon 1991. At present Anne Tihon is preparing an edition and translation of Book IV.

[45] Mogenet & Tihon 1985, pp. 97 and 162.

[46] Mogenet & Tihon 1985, pp. 98–100 and 163–164.

[47] Mogenet & Tihon 1985, pp. 117–120 and 177–179. See the commentary on pp. 276 ff. for an explanation of the concepts local time and seasonal hours.

[48] The Small Commentary is edited and translated in Tihon 1978.

[49] Tihon 1978, pp. 216–218 and 309–310.

3.2.2 Preliminaries

3.2.2.1 Description of the table

The manuscript *Leiden BPG 78* contains a copy of the *Handy Tables* on folios 50^v–155^r. This copy is written in uncial Greek and can be dated to the early 9th century.[50] On folios 75^r–76^v we find a table entitled "Ορθῆς σφαίρας σὺν μεσουρανήματι πανταχῆ" (literally: [Ascensions] of the right sphere with culmination everywhere; see Plate 3.1). Each page displays seven columns: one for the argument and two columns with tabular values for each of three zodiacal signs, starting with Capricorn on folio 75^r. The argument is the solar position, which runs from 1° to 30° with steps of 1°. For each zodiacal sign the first column (headed "ascensions", ἀναφοράι) displays the "normed right ascension" (i.e. the right ascension with the winter solstice as reference point instead of the vernal equinox), the second column (headed "sixtieths of hours", ὡρῶν ἑξηκοστὰ) the equation of time.[51]

The right ascension values are given to minutes. Theon described their mathematical structure in his *Great Commentary on the Handy Tables* (see Section 3.2.2.2 below). The equation of time is tabulated in minutes and seconds of an hour. It has a maximum $0;33,23^h$ for 18° Aquarius, a secondary minimum $0;6,12^h$ for 0° Gemini, a secondary maximum $0;16,21^h$ for 9° Leo, and a minimum $0;0,0^h$ for 0–3° Scorpio. From the relative positions of the extremes it follows that the tabular entries must be added to the true solar time to obtain mean solar time. This implies that they were probably computed according to a variant of formula 3.5 or 3.6, namely $E_h^{HT}(\lambda) = \frac{1}{D}(\alpha(\lambda) - \lambda + q(\lambda) + c)$ or $\bar{E}_h^{HT}(\bar{\lambda}) = \frac{1}{D}(\alpha(\bar{\lambda} + \bar{q}(\bar{\lambda})) - \bar{\lambda} + c)$ depending on the independent variable.

In the subsequent sections I will describe the available specific historical information concerning the equation of time as used by Ptolemy and will then carry out a detailed analysis to recover the mathematical structure of the table in the *Handy Tables*. We will see that none of the primary sources mentioned in Section 3.2.2.1 gives correct detailed information about the method according to which Ptolemy computed his table. Although various modern authors have described the equation of time in Ptolemy's works extensively,[52] none of them has made a serious attempt to recover the method of computation of the table.

3.2.2.2 Specific historical information

In the sources mentioned in Section 3.2.1 we find the following information with regard to the independent variable and the underlying parameters of Ptolemy's table for the equation of time:

[50] A description of the manuscript Leiden BPG 78 can be found in Tihon 1978, pp. 105–106. A list of all tables that the manuscript contains is presented in Tihon 1992, pp. 58–61.

[51] The complete table is transcribed in Stahlman 1959, pp. 206–209. Since Stahlman did not consult the Leiden manuscript, there may be incidental differences. In general there are so few differences between the copies of the equation of time table in the four uncial manuscripts of the *Handy Tables* that the following analysis would be practically identical had we started from one of the other copies. In Section 3.2.3.6 I will check the results of my final recomputation against five other manuscripts.

[52] See especially Rome 1939 and Neugebauer 1975, vol. 1, pp. 61–68 and vol. 2, pp. 984–986. See also Rome's notes on the table in Rome 1931–1943, vol. 3, pp. 922–924.

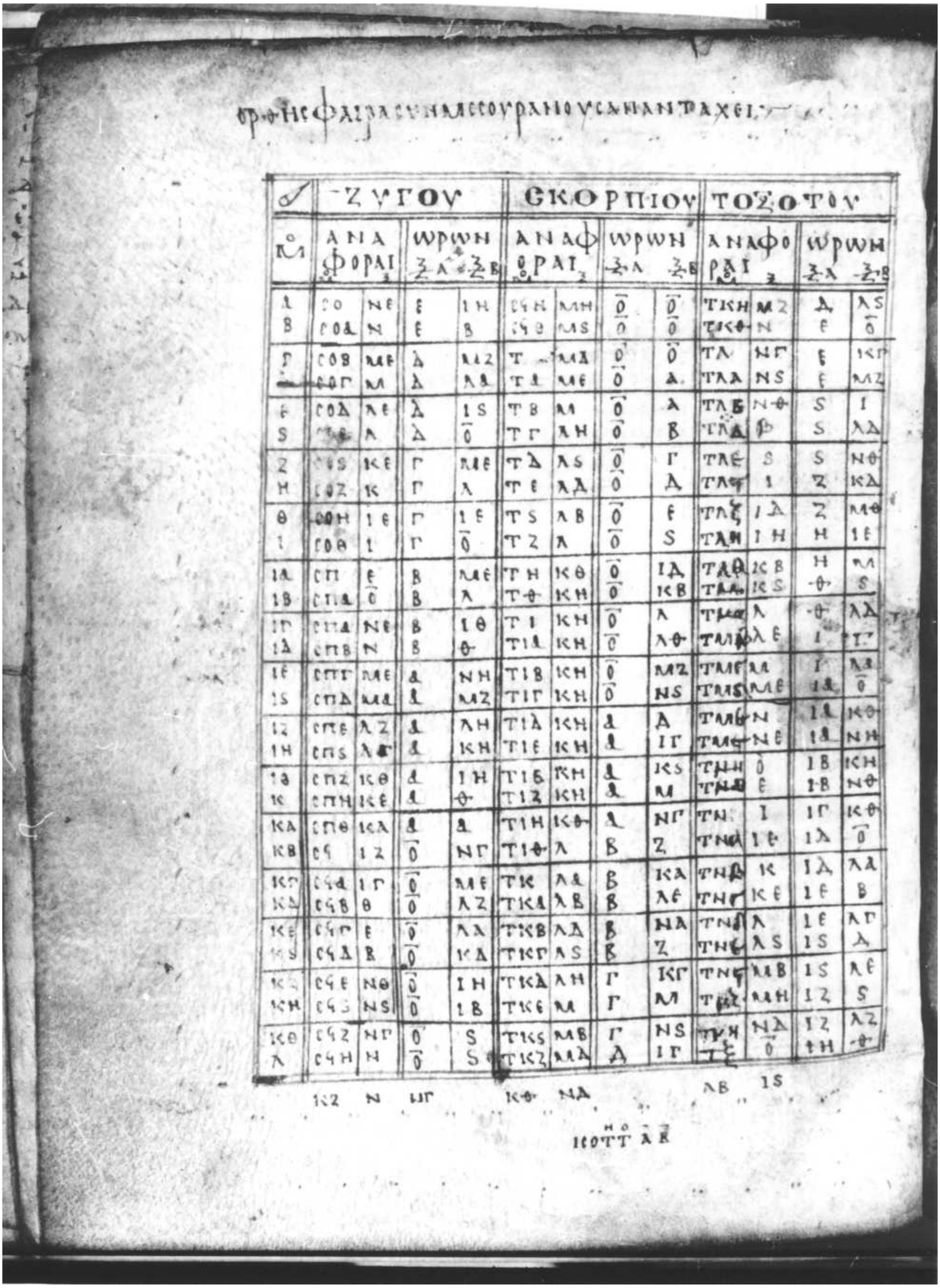

Plate 3.1: Fragment of the table for the equation of time in Leiden BPG 78 (folio 76ᵛ)

- The independent variable is the true solar position. This can be concluded from all relevant sources:

  In Ptolemy's *Introduction to the Handy Tables* the author first explains how the true solar position can be computed using the mean motion and equation tables and then states: "the degree (μοῖρα) on which the thus obtained number falls, that one we shall use separately (ἐκθησόμεθα) for the distinction of the times, which consists of three differences" (meant is the conversion from local true solar time given in seasonal hours to Alexandria mean solar time, the third step of which is the application of the equation of time; see Section 3.2.1).[53] Further on when he describes the conversion from true solar time to mean solar time (i.e. the third step mentioned above) Ptolemy writes: "The seconds of an equinoctial hour adjacent to the solar position (ἡ ἡλιακὴ μοῖρα) in the table of the right ascensions, if we want to take the equal [days] from the apparent days, we always add them [i.e. the seconds of an equinoctial hour] to the apparent hours; if [we want] the apparent [days] from the equal [days], we always subtract them from the equal [hours]."[54]

  In the *Great Commentary on the Handy Tables* Theon leaves no doubt about the independent variable of the table for the equation of time. First he describes how the true solar position can be determined from the mean position by adding or subtracting the solar equation. The result is called "the exact position of the sun" (ἡ ἀκριβὴς τοῦ ἡλίου ἐποχή).[55] Further on he applies the equation of time to convert true to mean solar time: "again we enter the right ascension table with the exact degree of the sun (ἡ ἀκριβὴς τοῦ ἡλίου μοῖρα)".[56] Also from the description of the right ascension table and from the explanation of the computation of the equation of time it is clear that the independent variable of the table is taken to be the true solar longitude.[57]

  The terminology in the *Small Commentary on the Handy Tables* is very similar to that in the *Great Commentary*. By adding or subtracting the solar equation, one obtains "the exact position of the sun" (ἡ ἀκριβὴς τοῦ ἡλίου ἐποχή) from the mean solar motion.[58] To apply the equation of time the right ascension table is entered with the degree of the sun (ἡ τοῦ ἡλίου μοῖρα). Also the numerical example is in accordance with the assumption that the independent variable of the table for the equation of time is the true solar position.[59]

- For the obliquity of the ecliptic ε Ptolemy uses the value 23;51,20 throughout the *Almagest*. Since the equation of time depends on the obliquity through the right ascension, the right ascension in its turn through the declination, we have to consider the declination and right ascension tables in Ptolemy's works as well. The declination

[53] Heiberg 1907, p. 161, lines 12–15; Halma 1822–1825, vol. 1, p. 3, lines 11-14. All English translations are my own and are in many cases based on the French translations by Halma, Mogenet, and Tihon.
[54] Heiberg 1907, p. 162, line 23–p. 163, line 5; Halma 1822–1825, vol. 1, p. 5, lines 11–18.
[55] Mogenet & Tihon, p. 115, lines 17–18; French translation on pp. 175–176.
[56] Mogenet & Tihon 1985, p. 120, lines 8–9; French translation on pp. 178–179.
[57] Mogenet & Tihon 1985, pp. 97–100; French translation on pp. 162–164.
[58] Tihon 1978, pp. 207–210; French translation on pp. 304–306.
[59] Tihon 1978, pp. 216–218; French translation on pp. 309–310.

table in the *Almagest*[60] is obviously based on ε = 23;51,20, since this value appears for argument 90. Furthermore, the right ascension values for ecliptical arcs of 10 degrees presented in Chapter 16 of Book I are correct for ε = 23;51,20.[61] Although the declination table in the *Handy Tables*[62] gives values to minutes only, I found that it is based on obliquity 23;51,20 as well.[63]

In his *Great Commentary* Theon indicates correctly that the right ascension table in the *Handy Tables* was computed from the values given in the *Almagest* by means of a particular type of linear interpolation which I will call "distributed linear interpolation".[64] As usual the difference between two consecutive nodes was distributed over the tabular differences in such a way that they differ at most by a single unit in their final sexagesimal digit.[65] But instead of the tabular differences being distributed evenly between the two nodes, the larger differences were placed closer to the nearest solstice, the smaller differences, closer to the nearest equinox. Such a distribution is in accordance with the fact that the right ascension increases faster in the neighbourhood of the solstices. In fact it turns out that for the right ascension this type of linear interpolation is more accurate than the usual type which involves even distribution of the tabular differences.

- In the *Almagest* Ptolemy finds the value 2;29$\frac{1}{2}$ for the solar eccentricity e, which he rounds to 2;30.[66] He uses the rounded value in his worked examples,[67] but it is doubtful whether it also underlies his table for the solar equation as a function of the mean solar longitude.[68] In the *Great Commentary* Theon states that the solar equation table in the *Handy Tables*[69] was derived from the one in the *Almagest* using the same type of linear interpolation which was used for the computation of the right ascension.[70] In fact the interpolation pattern concerned can be recognized in nearly every internodal block of the table. However, the nodal values $q(6), q(12), q(18), \ldots, q(174)$ differ from the corresponding values in the *Almagest* in 6 out of 29 cases, and fit the rounded eccentricity value 2;30 significantly better than the *Almagest* values.[71] No information about the solar eccentricity can be found in the *Small Commentary*.
- In all sources that I consulted the solar apogee is stated to be in 5°30′ Gemini, i.e. λ_A = 65;30.

60 Heiberg 1898–1903, vol. 1, p. 80–81; Toomer 1984, p. 72.
61 Heiberg 1898–1903, vol. 1, p. 85; Toomer 1984, p. 74.
62 Leiden BPG 78, f. 97^r; Stahlman 1959, p. 260.
63 Unpublished result.
64 See Rome 1939, p. 219; Mogenet & Tihon 1985, pp. 97 and 162.
65 Thus the nodes $\alpha(10)$ = 10;55 and $\alpha(20)$ = 21;42 yield three tabular differences of 1;4, seven of 1;5.
66 Heiberg 1898–1903, vol. 1, p. 236; Toomer 1984, p. 155.
67 Heiberg 1898–1903, vol. 1, p. 240 ff.; Toomer 1984, p. 157 ff.
68 Heiberg 1898–1903, vol. 1, p. 253; Toomer 1984, p. 167. Assuming that Ptolemy calculated every single tabular entry using the correct formula which he demonstrates in his worked examples, e = 2;29$\frac{1}{2}$ fits the table better than e = 2;30; see van Dalen 1988, pp. 10–12. See also Van Brummelen 1993, pp. 149–150.
69 Leiden BPG 78, ff. 94^r–96^v; Stahlman 1959, pp. 249–254.
70 Mogenet & Tihon 1985, pp. 107–109 and 169–171.
71 See also the commentary on this table in Mogenet & Tihon 1985, pp. 253–264.

- No explicit information about the epoch constant can be found in either of our sources. The following, however, can be said about the epochs of the planetary motions in Ptolemy's works:

  In the *Almagest* the epoch of the mean motion tables is the Era Nabonassar. Ptolemy gives $\bar{\lambda}_0 = 330;45$ and $\lambda_0 = 333;8$ as epoch values for the mean and true solar longitudes,[72] whence $\bar{\lambda}_0 - \alpha(\lambda_0) \approx 330;45 - 335;8 = -4;23°$. Since the equation of time at epoch was close to minimum, an epoch constant c equal to 4;23 will lead to equation of time values which must (nearly) always be *added* to the mean solar time to obtain true solar time. The table that is found in the papyrus London 1278 has this property.[73]

  In the *Handy Tables* the epoch had been shifted to the Era Philip. Thus we have $\bar{\lambda}_0 = 227;40$ (from the solar mean motion table), $\lambda_0 = \bar{\lambda}_0 - \bar{q}(\bar{\lambda}_0 - 66) \approx 226;54$ (using the solar equation table), and $\bar{\lambda}_0 - \alpha(\lambda_0) \approx 227;40 - 224;22 = 3;18$ (using the right ascension table).[74] Since in this case the equation of time at epoch was close to maximum, an epoch constant c equal to $-3;18$ leads to equation of time values of which the absolute value must always be *subtracted* from mean solar time to obtain true solar time.[75] The equation of time in the *Handy Tables* in fact has this property.

- In all used sources a division by 15 ($\bar{\iota}\,\bar{\epsilon}$) is said to be used to convert equinoctial degrees into hours. In the *Almagest* it is mentioned that the mean solar day equals $360°+0;59°$ approximately,[76] but this fact does not seem to have been utilized to carry out a more accurate conversion. Thus we expect $D = 15$.

In his article *Le problème de l'équation du temps chez Ptolémée* Rome describes the usage of the equation of time table in the *Handy Tables*, but does not add any information about the mathematical structure of the table to what is found in the above-mentioned primary sources.[77] Rome devotes a section to the astronomer Serapion,[78] who according to Theon indicated that a correction is necessary to compensate for the fact that the minimum equation of time occurs at 0° Scorpio, whereas the solar position at the epoch 1 Philip was 17° Scorpio.[79] Since several sources mention a geographer Serapion who lived before Ptolemy,[80] we have to consider the possibility that Ptolemy adopted the table for the equation of time from an earlier work. An extensive discussion of this matter is beyond the scope of this case study, but an interesting remark concerning the error in the solar

[72] Heiberg 1898–1903, vol. 1, pp. 256–257 and 263; Toomer 1984, pp. 168–169 and 172.

[73] Neugebauer 1958, pp. 97–102 and 109–111. See also Section 3.2.5.

[74] See Leiden BPG 78, ff. 97^r, 94^r–96^v, and 75^r–76^v; Stahlman 1959, pp. 243, 249–254, and 206–209.

[75] As was mentioned before, Ptolemy and mediaeval Islamic astronomers did not make use of negative numbers. Instead they always tabulated the absolute value of quantities like the equation of time and indicated in the margin of the table or in the explanatory text in which cases the quantity had to be added or subtracted.

[76] Heiberg 1898–1903, vol. 1, p. 258; Toomer 1984, p. 170.

[77] Rome 1939, pp. 217–218.

[78] Rome 1939, pp. 223–224.

[79] See Mogenet & Tihon 1985, pp. 123, lines 9–14; French translation on p. 182.

[80] Cf. *Pauly*, 2nd series, vol. 2, cols 1666-1667. Neugebauer suggests that the Serapion mentioned by Theon can be identified with the Alexandrian astrologer of the same name listed in *Pauly* as number 1; see Neugebauer 1958, pp. 110–111. See also Tihon 1992, pp. 74–75.

position at epoch will be made in Section 3.2.5.

In his edition of Theon's *Commentary on the Almagest* Rome makes some remarks about the equation of time table in the *Handy Tables* in the footnotes of the section on the "inequality of the days".[81] He investigates the tabular differences to check Ptolemy's statement that the maximum variation in the value of the equation of time is 31″ per day and finds that only one difference of 32″ occurs.[82] He also mentions that if Ptolemy had actually used the procedure described by Theon in his *Great Commentary* (see above), all tabular values should be multiples of 4″, which is not the case.

3.2.2.3 Rounding

Neither in the *Almagest* and the *Handy Tables* nor in Theon's commentaries on these works can explicit information about Ptolemy's method of rounding be found.[83] I investigated some of Ptolemy's tables and found that modern rounding was used in the computation of the right ascension values in the *Almagest* and the declination table in the *Handy Tables*. The recomputation of the solar equation tables in both sources is problematical,[84] but nevertheless the use of modern rounding can be shown to be more probable than upward rounding or truncation. From recent research by Glen Van Brummelen it can be concluded that the calculation of the table of chords in the *Almagest* and its sixtieths involves modern rounding as well.[85] No conclusion about the rounding method used could be drawn in the case of the declination table in the *Almagest*, which is highly inaccurate. As was explained above, the right ascension table in the *Handy Tables* is derived from the values given in the *Almagest* by means of a type of linear interpolation which does not involve rounding (cf. Section 3.2.2.2). Dr. Alexander Jones kindly drew my attention to the fact that many of the planetary mean motion tables in the *Handy Tables* were calculated from those in the *Almagest* by rounding the tabular values in the modern way to a single sexagesimal fractional digit. By applying the "least number of errors criterion" (see Section 2.5), I verified that the mean motion tables not established in this way, also involve modern rounding.

Since in several of the above-mentioned tables the use of modern rounding could be established and in none of them are obvious traces of upward rounding or truncation present, I will in principle assume that Ptolemy made use of modern rounding for the computation of his table for the equation of time. At various critical stages of the analysis in Section 3.2.3 I checked the possibility of other types of rounding, but in no case did this lead to better results. Therefore these results will not be mentioned.

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[81] Rome 1931–1943, vol. 3, pp. 922–924.

[82] The irregular differences that Rome obtains starting from 5° Capricorn are a result of two errors in the tabular values given by Halma on pp. 148–155 of Halma 1822–1825, vol. 1, namely $T(247) = 0;21;41$ (should be 0;21;45) and $T(250) = 0;23,16$ (should be 0;23,13). The correct values can be found in the manuscripts Leiden BPG 78 and Vatican gr. 1291.

[83] Cf. Section 1.1.2.

[84] See also Section 3.2.2.2.

[85] See Van Brummelen 1993, pp. 60–73.

true solar long.	tabular differ-ences	true solar long.	tabular differ-ences	true solar long.	tabular differ-ences	true solar long.	tabular differ-ences
36	−0; 0,14	48	−0; 0, 6	60	0; 0, 8	72	0; 0, 9
37	−0; 0,15	49	−0; 0, 6	61	−0; 0, 1	73	0; 0, 9
38	−0; 0,14	50	−0; 0, 5	62	0; 0, 4	74	0; 0, 9
39	−0; 0,15	51	−0; 0, 6	63	0; 0, 3	75	0; 0, 9
40	−0; 0,14	52	−0; 0, 5	64	0; 0, 4	76	0; 0, 9
41	−0; 0,13	53	−0; 0, 6	65	0; 0, 4	77	0; 0, 9
42	−0; 0,12	54	−0; 0, 3	66	0; 0, 5	78	0; 0,11
43	−0; 0,11	55	−0; 0, 4	67	0; 0, 5	79	0; 0,11
44	−0; 0,10	56	−0; 0, 3	68	0; 0, 6	80	0; 0,11
45	−0; 0,10	57	−0; 0, 4	69	0; 0, 5	81	0; 0,11
46	−0; 0,10	58	−0; 0, 3	70	0; 0, 6	82	0; 0,12
47	−0; 0,10	59	−0; 0, 4	71	0; 0, 5	83	0; 0,11

Table 3.1: First differences of Ptolemy's equation of time

3.2.3 Technical Analysis

3.2.3.1 Interpolation

First I investigated the tabular differences of Ptolemy's table for the equation of time. A typical sample is found in Table 3.1.[86] It can be noted that for most multiples of 6° of the argument there are obvious jumps in the first differences.[87] Furthermore in nearly all cases the six differences between tabular values for consecutive multiples of 6° differ by 1 second at most. Although the patterns in these groups of differences show little consistency, we conclude that Ptolemy used linear interpolation within intervals of 6°. Stahlman mentions the possibility of interpolation within intervals of 3 degrees.[88] However, it can be shown that the tabular values for arguments $(3 + 6n)^\circ$ much more probably were computed by means of linear interpolation within intervals of 6 degrees than by the (less exact) method according to which the values for multiples of 6 degrees

[86] The irregular differences for arguments 60 and 61 result from a scribal error: $T(61) = 0;6,20$ must be corrected to 0;6,16, which is found in two other manuscripts that I consulted. We will see later that $T(42) = 0;8,10$ can be corrected to 0;8,8. This makes the interpolation pattern even more regular. See Section 3.2.3.6 for a more extensive discussion of possible scribal errors.

[87] In the displayed part of the table the jumps for multiples of 12° are significantly larger than those for arguments $(6 + 12n)^\circ$. This is an accidental circumstance which is a result of the method by which the underlying right ascension will be shown to have been computed.

[88] Stahlman 1959, pp. 43–44. Stahlman's argument that the solar equation in the *Almagest* is partly tabulated for multiples of 3 degrees, partly for multiples of 6 degrees, will turn out to be irrelevant, since the solar equation table in the *Almagest* has the mean solar anomaly as its independent variable, whereas the table for the equation of time in the *Handy Tables* is a function of the true solar longitude.

true solar long.	equation of time	error	true solar long.	equation of time	error
6	0;20,33	+3	186	0; 4, 0	+2
12	0;18, 4	+4	192	0; 2,30	−3
18	0;15,38	+4	198	0; 1,28	+5
24	0;13,24	+7	204	0; 0,37	+7
30	0;11,13	−1	210	0; 0, 0	
36	0; 9,35	+7	216	0; 0, 2	+7
42	0; 8,10	+8	222	0; 0,22	+6
48	0; 7, 7	+6	228	0; 1,13	+6
54	0; 6,33	+9	234	0; 2,35	+9
60	0; 6,12	−1	240	0; 4,13	
66	0; 6,34	+7	246	0; 6,34	+7
72	0; 7, 6	+4	252	0; 9, 6	+4
78	0; 8, 0	+3	258	0;11,58	+3
84	0; 9, 7	+1	264	0;15, 2	+2
90	0;10,23		270	0;18, 9	
96	0;11,42	−1	276	0;21,15	−1
102	0;12,57	−3	282	0;24,11	−2
108	0;14, 3	−4	288	0;26,50	−4
114	0;14,53	−7	294	0;29, 4	−8
120	0;15,37	+1	300	0;31, 4	+1
126	0;16, 3	+12	306	0;32,15	−9
132	0;16, 2	+16	312	0;33, 6	−7
138	0;15,18		318	0;33,23	−5
144	0;14,27	−4	324	0;33, 5	−7
150	0;13,27	+1	330	0;32,26	
156	0;11,59	−7	336	0;31, 6	−6
162	0;10,31	−3	342	0;29,30	−4
168	0; 8,52	−4	348	0;27,33	−4
174	0; 7,11	−3	354	0;25,21	−3
180	0; 5,33		360	0;23, 0	

Table 3.2: Preliminary recomputation of Ptolemy's equation of time

will be shown to have been determined. It can be noted that the interpolation method occurring in the tables for the right ascension and the solar equation in the *Handy Tables* was not applied to the table for the equation of time.[89] A completely irregular pattern (tabular differences $+6, +6, +6, -6, -6, -7''$) occurs around the secondary maximum at 9° Leo (which is the only extreme value which does not occur at a multiple of 6 degrees). Later we will see that other values in the neighbourhood of this maximum are corrupt as well.

I will assume that the tabular values for multiples of 6 degrees were calculated independently and will refer to these values as the *nodes*. They are displayed in the second and fifth columns of Table 3.2, where the argument (in the first and fourth columns) is degrees of the ecliptic reckoned from the vernal point.[90] I corrected two obvious scribal errors in the nodes: $T(108) = 0;14,3$ and $T(210) = 0;0,6$. A systematic discussion of scribal errors is presented in Section 3.2.3.6. Only the nodal values will be used in the mathematical analysis of the table.

3.2.3.2 Independent variable

I performed least squares estimations for the parameter values underlying Ptolemy's equation of time table for both possibilities of the independent variable. If we assume that the independent variable is the mean solar longitude, the minimum possible standard deviation of the residuals is $21''$ if all nodes are used, $24''$ if only tabular values for multiples of 30° are used. If we assume that the independent variable is the true solar longitude, the minimum possible standard deviations of the residuals are $4''$ and $20'''$ (thirds) respectively.[91] We conclude that the true solar longitude is the independent variable of the table, which is in agreement with the information found in the historical sources dealing with the equation of time as used by Ptolemy (see Section 3.2.2.2). Our conclusion will be confirmed by the recomputations of the extracted solar equation and right ascension tables (see below). Hereafter I will call the tabular values for multiples of 30° *supernodes*. The large difference between the minimum possible standard deviation if all nodes are used and if only supernodes are used will be explained below.

3.2.3.3 Preliminary estimates of the parameters

To get a first impression of the underlying parameter values and the accuracy of the table, we compute confidence intervals for the parameters using a least squares estimation as explained in Section 2.4. Since we have already seen that the supernodes yield a much

[89] Cf. Section 3.2.2.2.

[90] In Leiden BPG 78 the tabular values for the nodes 342, 360 and for several other arguments are illegible. Therefore I used the values found in another early manuscript, Vatican gr. 1291, which was used by Stahlman for his edition of the Handy Tables in Stahlman 1959. In the Vatican manuscript the table for the right ascension and the equation of time is found on folios 22^r–23^v.

[91] These results are not affected by the choice of the conversion factor (15 or 15;2,28). Bear in mind that the tabular errors of a correct tabulation to seconds have a standard deviation of approximately $17'''$.

better fit than the ordinary nodes, we will base our estimations on the supernodes only. For conversion factor 15 the results are as follows:

parameter	*95 % confidence interval*
obliquity of the ecliptic ε	⟨ 23;51,50 , 23;52,26 ⟩
solar eccentricity e	⟨ 2;29,51 , 2;30, 0 ⟩
solar apogee λ_A	⟨ 65;57,25 , 66; 0,38 ⟩
epoch constant c	⟨ 3;34, 3 , 3;34, 9 ⟩

For conversion factor 15;2,28 we find as confidence intervals ⟨23;53,45 , 23;54,20⟩, ⟨2;30,16 , 2;30,24⟩, ⟨65;57,28 , 66;0,34⟩, and ⟨3;34,39 , 3;34,44⟩ respectively. For conversion factor 15 the results correspond somewhat better to the historically plausible values $\varepsilon = 23;51,20$ (Ptolemy's obliquity of the ecliptic), $e = 2;30$ (Ptolemy's solar eccentricity) and $\lambda_A = 66;0$ (rounded from Ptolemy's apogee value 65;30). The epoch constant c was apparently chosen in such a way that the minimum tabular value became 0;0,0. The fact that the minimum equation of time occurs for $\lambda = 210$, whereas the solar position at the epoch 1 Nabonassar was 17° Scorpio, explains why the estimated epoch constant is somewhat different from the constant predicted in Section 3.2.2.2.[92]

I recomputed Ptolemy's table for the equation of time using conversion factor 15, the above-mentioned historically plausible values for ε, e, and λ_A, and $c = 3;34,6$ (least squares estimate of c for conversion factor 15). The errors in the final sexagesimal digit (Ptolemy's table minus recomputation) are shown in the third and sixth columns of Table 3.2. At first sight the results appear to be disastrous. However, it can be noted that in fact the supernodal values are much more accurate than the ordinary nodes (as could be expected because of the minimum standard deviations found above). Furthermore, the errors in between the supernodes are nearly all positive in the first and third quadrants, but negative in the second and fourth. The symmetry in the right ascension and the solar equation mentioned in Section 3.1.3 suggests that the errors were caused by inaccuracies in the underlying right ascension rather than in the solar equation.

Note that obvious outliers (i.e. errors which do not fit into the overall error pattern) are found for arguments $\lambda = 126$, 132, 138 and 192. It turned out to be impossible to correct these errors on the basis of the interpolation pattern or by consulting other manuscripts containing the table for the equation of time. The possibility of scribal errors in the Greek transmission will be investigated after establishing the method according to which the table was calculated. The outliers will be disregarded in the estimation of the underlying parameter values.

3.2.3.4 The extracted solar equation

Using formula (3.11) we can extract the solar equation underlying Ptolemy's table for the equation of time as a function of the true solar longitude. The only condition for the

[92] Cf. the remarks about Serapion at the end of Section 3.2.2.2 and about the epoch constant of the equation of time as tabulated by Ptolemy after the conclusions of Section 3.2.5. The difference of $16'$ between the estimated and predicted epoch constants corresponds to the systematic error of $1'4''$ in the equation of time values in the *Handy Tables* as noted by Serapion.

validity of the extraction is that the used tabulations of the right ascension and the solar equation satisfy the symmetry relations mentioned in Section 3.1.3. This can safely be assumed since Ptolemy utilizes the symmetry in the *Almagest*.

The result of the extraction when the conversion factor D is taken equal to 15 can be found in the second column of Table 3.3 (the first column displays the argument, the true solar longitude). Note that the values for $\lambda = 12$, 126, 132 and 138 are probably outliers, since they were derived from outlying values of the equation of time. If the tabular values of the equation of time were correct,[93] they would only contain a rounding error of at most $30'''$ in absolute value. Consequently the extracted solar equation would have a maximum absolute error of $7\frac{1}{2}''$ and the standard deviation of its errors would be approximately $3''$.[94] Hereafter, when I speak about the errors in an extracted table, I will mean the differences between the extracted table and a particular recomputation which are larger than $7\frac{1}{2}''$ in absolute value.

First note that the fact that the extracted solar equation equals 0;0,0 for solar longitude 66° (and to a lesser extent also the fact that the maximum 2;23,22,30 occurs at 156° and that the table is almost symmetric around 66° and 156°) confirms our preliminary estimate $66°0'$ for the solar apogee.[95] A least squares estimation using all extracted values minus the four above-mentioned outliers yields a minimum possible standard deviation of $7''$ and 95 % confidence intervals ⟨2;29,52 , 2;30,0⟩ for the eccentricity e, ⟨65;58,30 , 66;1,19⟩ for the apogee λ_A. If we extract the solar equation using $D = 15;2,28$ the minimum possible standard deviation is the same, but the 95 % confidence intervals are ⟨2;30,17 , 2;30,25⟩ and ⟨65;58,30 , 66;1,19⟩ respectively.

Independent of the value used for the conversion factor we can conclude that Ptolemy calculated the equation of time using the value $66°0'$ for the solar apogee. In fact, in both cases this value lies in the middle of the confidence interval, whereas the value mentioned in all historical sources ($65°30'$) is far removed from the interval. Below I will argue that the value $66°0'$ is historically plausible, although not attested. Since for $D = 15;2,28$ the confidence interval for the solar eccentricity does not contain any historically plausible (i.e. either attested or round) value, we conclude that the conversion factor used is 15, the solar eccentricity 2;30.[96]

In the *Almagest* and the *Handy Tables* Ptolemy gives tables for the solar equation

[93]Remember that a tabular value is said to be *correct* if it corresponds in all sexagesimal places to a value accurately computed according to a presumed algorithm and rounded according to a presumed rounding method (see Section 1.1.3). A tabular value which is not correct is said to be *in error*. If no mention of the used rounding method is made, I assume that the rounding is performed in the modern way.

[94]Here I apply the assumption that the tabular errors of a correct tabulation to seconds of the equation of time have approximately a uniform distribution (see Section 1.2.4). The errors in the extracted tables are then equal to the sum of two uniformly distributed variables and therefore have approximately a triangular distribution.

[95]For $\lambda_A = 65;30$ we would have $q(66) \approx$ 0;1,15, which is significantly different from 0;0,0 since the expected maximum error is $7\frac{1}{2}''$.

[96]In fact this choice of D and e leads to only 3 errors in the recomputation, whereas all other historically plausible choices (involving $e = 2;29,30$ and $D = 15;2,28$) yield at least 18 errors in 26 extracted values.

true solar long.	extracted solar equation	recomputed solar equation	error
6	−2; 4, 7,30	−2; 4, 4,33	−0; 0, 2,57
12	−1;56,45, 0	−1;55,54,18	−0; 0,50,42
18	−1;46,15, 0	−1;46,27,53	0; 0,12,53
24	−1;35,52,30	−1;35,51,30	−0; 0, 1, 0
30	−1;24, 7,30	−1;24,12, 9	0; 0, 4,39
36	−1;11,37,30	−1;11,37,30	0; 0, 0, 0
42	−0;58,30, 0	−0;58,15,49	−0; 0,14,11
48	−0;44,15, 0	−0;44,15,53	0; 0, 0,53
54	−0;29,45, 0	−0;29,46,54	0; 0, 1,54
60	−0;14,52,30	−0;14,58,22	0; 0, 5,52
66	0; 0, 0, 0	0; 0, 0, 0	0; 0, 0, 0
72	0;15, 0, 0	0;14,58,22	0; 0, 1,38
78	0;29,45, 0	0;29,46,54	−0; 0, 1,54
84	0;44,22,30	0;44,15,53	0; 0, 6,37
90	0;58,15, 0	0;58,15,49	−0; 0, 0,49
96	1;11,37,30	1;11,37,30	0; 0, 0, 0
102	1;24,15, 0	1;24,12, 9	0; 0, 2,51
108	1;35,52,30	1;35,51,30	0; 0, 1, 0
114	1;46,22,30	1;46,27,53	−0; 0, 5,23
120	1;55,52,30	1;55,54,18	−0; 0, 1,48
126	2; 1,30, 0	2; 4, 4,33	−0; 2,34,33
132	2; 8, 0, 0	2;10,53,15	−0; 2,53,15
138	2;15,37,30	2;16,15,52	−0; 0,38,22
144	2;19,45, 0	2;20, 8,53	−0; 0,23,53
150	2;22,22,30	2;22,29,44	−0; 0, 7,14
156	2;23,22,30	2;23,16,51	0; 0, 5,39
162	2;22,22,30	2;22,29,44	−0; 0, 7,14
168	2;20, 7,30	2;20, 8,53	−0; 0, 1,23
174	2;16,15, 0	2;16,15,52	−0; 0, 0,52
180	2;10,52,30	2;10,53,15	−0; 0, 0,45

Table 3.3: Recomputation of the extracted solar equation

as a function of the mean solar anomaly.[97] In both cases the tabular values are given to minutes. To compute the table for the equation of time, however, Ptolemy needed the solar equation as a function of the true solar anomaly. In his *Great Commentary* Theon suggests that Ptolemy used inverse linear interpolation in the *Almagest* table to obtain the desired values.[98] However, it can be shown that inverse linear interpolation in any solar equation table with values to minutes is not sufficiently accurate to produce the extracted solar equation table.[99] Thus we conclude that Ptolemy had in fact independently computed values for the solar equation as a function of the true solar longitude.

Note that such solar equation values are not very difficult to obtain. For each argument Ptolemy had to look up a value in the table of chords, divide that value by 24 (multiply by 0;2,30), and perform an inverse linear interpolation in the table of chords. The algorithm (equivalent to our formula 3.3) is indicated in the *Almagest* section on the solar model[100] and a worked example is given in Ptolemy's calculation of the mean solar position at an observed autumnal equinox.[101] It can be checked that the table of chords in the *Almagest*[102] is sufficiently accurate to produce the solar equation values which we extracted from the table for the equation of time.[103]

The computation of the solar equation values is equally easy for $\lambda_A = 65°30'$ and $\lambda_A = 66°0'$; one only has to start with a different value from the table of chords. The degree of difficulty of the inverse interpolation in the final step of the computation is not affected by the choice of λ_A. Ptolemy's use of $\lambda_A = 66°0'$ for the equation of time may be explained from the existence of a tabulation for the solar equation as a function of the true solar anomaly a ($a = \lambda - \lambda_A$). If such a tabulation only gave the solar equation for integer values of the anomaly (or for multiples of 3° or 6°), the use of $\lambda_A = 66°0'$ instead of $\lambda_A = 65°30'$ would avoid the need for interpolation. We can assume that Ptolemy was aware that the use of $\lambda_A = 66°0'$ instead of $65°30'$ only introduces negligible errors (the largest of which amounts to 5 seconds of time).

Recomputed solar equation values for λ_A = 66;0 and e = 2;30 are displayed in the third column of Table 3.3, differences between the extracted solar equation and the recomputation in the fourth column. Apart from the four outliers that we expected, there seem to be outliers for arguments 18, 42 and 144 as well. All other extracted values are correct, i.e. they differ from recomputed values by no more than $7\frac{1}{2}''$. Since the outliers do not satisfy the conditions for the least squares estimation, we must repeat the estimation

[97] Heiberg 1898–1903, vol. 1, p. 253; Toomer 1984, p. 167; Leiden BPG 78, ff. 94^r–96^v; Stahlman 1959, pp. 249–254.

[98] Mogenet & Tihon 1985, p. 99, line 16–p. 100, line 3; French translation on pp. 163–164.

[99] Inverse interpolation in the *Almagest* table leads to 19 errors (excluding the four outliers); in the *Handy Tables*, to 18. Inverse interpolation in a correct solar equation table with values to seconds for every degree of the mean solar anomaly would be accurate enough to produce the extracted solar equation, but there is no reason to believe that Ptolemy had such a table. Remember that in the context of an extracted table an error is a difference with a given recomputed value larger than $7\frac{1}{2}''$ in absolute value.

[100] Heiberg 1898–1903, vol. 1, pp. 242–243; Toomer 1984, p. 159.

[101] Heiberg 1898–1903, vol. 1, pp. 254–255; Toomer 1984, pp. 166 and 168.

[102] Heiberg 1898–1903, vol. 1, pp. 48–63; Toomer 1984, pp. 57–60.

[103] In fact, using Ptolemy's table of chords the solar equation can be computed with a maximum absolute error of $30'''$.

without the newly found outliers to obtain valid confidence intervals. If we use conversion factor 15 to extract the solar equation, the minimum possible standard deviation is $4''$ (slightly more than the standard deviation of the expected triangular distribution of the errors in the extracted values) and 95 % confidence intervals are $\langle$2;29,56 , 2;30,1$\rangle$ for the eccentricity and $\langle$65;58,46 , 66;0,19$\rangle$ for the apogee. For conversion factor 15;2,28 the minimum standard deviation and the confidence interval for the apogee are the same; a 95 % confidence interval for the eccentricity is given by $\langle$2;30,21 , 2;30,26$\rangle$. This confirms our conclusions.

Since the solar equation is symmetric around the apogee and perigee, we can compare the values for arguments $\lambda_A + 6k$ and $\lambda_A - 6k$ ($k = 1, 2, 3, \ldots, 14$) to obtain more insight into the origin of the errors. It turns out that for all seven outliers the symmetrically corresponding entry is correct. This implies that the errors were not made in the computation of the solar equation, but in a later stage of the determination of the equation of time values. The errors may be partly due to scribal mistakes.

3.2.3.5 The extracted right ascension

From the error pattern shown by our preliminary recomputation in Section 3.2.3.3 we concluded that rather large errors must be present in the right ascension values that were used for the computation of Ptolemy's table for the equation of time. We will now try to discover the origin of these errors by extracting the right ascension using formula (3.10). Here the main problem is that we need a value for the epoch constant c. As we have seen, no information about c can be found in the available historical sources. From the table for the equation of time itself it can be concluded that c was chosen in such a way that the minimum equation equals zero. Since the minimum is assumed for $\lambda = 210$ we expect

$$c \approx 210 - \alpha(210) + q(210 - 66) \approx 210 - 207;50,7 + 1;24,12 = 3;34,5.$$

On the other hand, the least squares estimate of c based on the supernodes only is $\hat{c}$ = 3;34,6 (see Section 3.2.3.3). Since it is impossible to decide on a value of c on the basis of the above information, I will examine the extracted right ascension for values of the epoch constant in the range 3;34,0 to 3;34,15.[104] If not mentioned separately, the results given below hold for all values of c in this range. Note that the extracted right ascension values for λ = 12, 126, 132 and 138 are probably outliers, since they were derived from outlying values of the equation of time.

First it can be noted that for the Ptolemaic obliquity value 23;51,20 and for all values of the epoch constant c in the above-mentioned range, the extracted right ascension values show the same error pattern as the equation of time: the supernodes contain small errors; in between the supernodes the errors are positive in the first quadrant and negative in the second. This situation cannot be remedied by varying the obliquity of the ecliptic: for values of c in the range 3;34,0 to 3;34,15 the minimum obtainable standard deviation of the errors in the extracted right ascension is $42''$, whereas the expected standard deviation

[104] Outside this range the number of errors in both the extracted right ascension and the equation of time is more than half the total number of tabular values.

of an extracted table is $3''$.[105] We conclude that the right ascension values were not all computed according to the precise formula (3.2) or an equivalent one.

As was noted before, in the *Almagest* Ptolemy only gives right ascension values for ecliptic arcs of $10°$.[106] In the *Handy Tables* the intermediate values were determined by means of so-called "distributed linear interpolation".[107] Therefore it seems plausible that the right ascension underlying the table for the equation of time also has interpolated values for non-multiples of $10°$. Since we assume that the equation of time was originally computed for intervals of $6°$, this would imply that only the extracted right ascension values for arguments which are common multiples of 6 and 10 are accurate. As we have seen, this is indeed the case.

The extracted right ascension values show little agreement with the values occurring in the *Handy Tables* in the column adjacent to the equation of time. In fact, the standard deviation of the differences is at least $79''$ for any value of c. Therefore I will consider seven historically plausible ways of computing the right ascension underlying Ptolemy's table for the equation of time, all based on obliquity 23;51,20. Whenever linear interpolation is involved the values for multiples of $10°$ are taken as nodes. The seven possibilities are:

A. Exact computation to minutes of every single right ascension value.

B. Exact computation to seconds of every single right ascension value.

C. Exact linear interpolation between correct nodes to seconds, followed by rounding to minutes.

D. Exact linear interpolation between correct nodes to seconds, followed by rounding to seconds.

E. Exact linear interpolation between the *Almagest* values without rounding (the resulting values are to seconds).

F. Exact linear interpolation between the *Almagest* values followed by rounding to minutes.

G. Distributed linear interpolation between the *Almagest* values as described by Theon (the resulting values occur in the *Handy Tables* in the column adjacent to the equation of time).

By "exact linear interpolation" I mean that the difference between every two consecutive nodes is distributed evenly between the internodal values. We have already noted that methods A, B and G are in poor agreement with the extracted right ascension values; we will use these methods mainly for comparison. Methods C and D are historically less plausible, since Ptolemy only gives right ascension values to minutes in the *Almagest* and *Handy Tables*.

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[105] See Section 3.2.3.4. For values of c outside the given range the minimum standard deviation is still significantly larger. If we set the obliquity to the Ptolemaic value 23;51,20 and vary the epoch constant, the minimum obtainable standard deviation is $70''$.

[106] See Section 3.2.2.2, footnote 61.

[107] See Section 3.2.2.2, footnote 64.

	$c = 3;34,0$			$c = 3;34,5$			$c = 3;34,10$			$c = 3;34,15$		
Type	n	μ	σ	n	μ	σ	n	μ	σ	n	μ	σ
A	18	+18	76	18	+13	75	19	+8	74	19	+3	74
B	22	+18	71	23	+13	70	24	+8	69	23	+3	69
C	10	+9	17	10	+4	15	12	−1	15	12	−6	16
D	13	+7	15	13	+2	13	15	−3	13	17	−8	15
E	4	+8	10	3	+3	7	6	−2	7	11	−7	9
F	10	+9	17	10	+4	15	12	−1	15	12	−6	16
G	17	+16	81	17	+11	80	19	+6	79	19	+1	79

Table 3.4: Error statistics for recomputations of the extracted right ascension

For each of the possibilities A to G and for the values 3;34,0, 3;34,5, 3;34,10, and 3;34,15 of the epoch constant, Table 3.4 displays the number of errors n,[108] the mean difference μ in seconds, and the standard deviation σ of the differences, also in seconds. Remember that in the case of a correct tabulation of the equation of time this standard deviation is expected to be approximately 3 seconds.[109] Since in all cases the four outliers were excluded, the total number of extracted right ascension values that were considered is 26.

From Table 3.4 we conclude without reservation that the right ascension underlying Ptolemy's table for the equation of time was computed according to method E, i.e. exact linear interpolation between the *Almagest* values without rounding. This follows both from the given numbers of errors and from the standard deviations. Even the methods which are similar to method E yield significantly worse results. Note that our conclusion is independent of the value of the epoch constant c, although the differences between the methods are most obvious in the middle of the considered range.

Assuming the use of method E we find that the number of errors in the extracted right ascension is minimized for $c = 3;34,7,30$. Since we cannot choose a historically more plausible value (for $c = 3;34$ the number of errors in the extracted right ascension is reasonably small, but the table for the equation of time itself has as many as 25 errors), we assume for the time being that the equation of time was computed using $c = 3;34,7,30$. Disregarding the four outliers, the extracted right ascension has only two errors for this value of the epoch constant: a difference of $16\frac{1}{2}''$ for argument 42 and a difference of $22\frac{1}{2}''$ for argument 144. Since these differences are significantly larger than all others, and because the extracted solar equation has errors for the same arguments, we conclude that both errors result from outliers in the table for the equation of time. The second column of Table 3.5 displays the extracted right ascension for epoch constant $c = 3;34,7,30$, the third column the right ascension recomputed according to method E and the fourth column the differences between the two. The argument in the first column is the true solar longitude. Note that all extracted right ascension values necessarily are multiples of $7\frac{1}{2}''$, the recomputed values, multiples of $12''$. Therefore there is only a limited number of

[108] Remember that in the context of an extracted table an error is a difference with a recomputed value larger than $7\frac{1}{2}''$ in absolute value.
[109] See Section 3.2.3.4.

true solar long.	extracted right ascension	recomputed right ascension	error
6	5;30, 0, 0	5;30, 0	0; 0, 0, 0
12	11; 0, 7,30	11; 1, 0	−0; 0,52,30
18	16;34, 7,30	16;34, 0	0; 0, 7,30
24	22;11, 0, 0	22;11, 0	0; 0, 0, 0
30	27;50, 0, 0	27;50, 0	0; 0, 0, 0
36	33;38, 0, 0	33;38, 0	0; 0, 0, 0
42	39;29,52,30	39;29,36	0; 0,16,30
48	45;28,22,30	45;28,24	−0; 0, 1,30
54	51;34,22,30	51;34,24	−0; 0, 1,30
60	57;44, 0, 0	57;44, 0	0; 0, 0, 0
66	64; 4,22,30	64; 4,24	−0; 0, 1,30
72	70;27,22,30	70;27,24	−0; 0, 1,30
78	76;55,37,30	76;55,36	0; 0, 1,30
84	83;27, 0, 0	83;27, 0	0; 0, 0, 0
90	89;59,52,30	90; 0, 0	−0; 0, 7,30
96	96;33, 0, 0	96;33, 0	0; 0, 0, 0
102	103; 4,22,30	103; 4,24	−0; 0, 1,30
108	109;32,30, 0	109;32,36	−0; 0, 6, 0
114	115;55,30, 0	115;55,36	−0; 0, 6, 0
120	122;16, 0, 0	122;16, 0	0; 0, 0, 0
126	128;28, 7,30	128;25,36	0; 2,31,30
132	134;34,22,30	134;31,36	0; 2,46,30
138	140;31, 0, 0	140;30,24	0; 0,36, 0
144	146;22,22,30	146;22, 0	0; 0,22,30
150	152;10, 0, 0	152;10, 0	0; 0, 0, 0
156	157;49, 0, 0	157;49, 0	0; 0, 0, 0
162	163;26, 0, 0	163;26, 0	0; 0, 0, 0
168	168;59, 0, 0	168;59, 0	0; 0, 0, 0
174	174;29,52,30	174;30, 0	−0; 0, 7,30
180	180; 0, 0, 0	180; 0, 0	0; 0, 0, 0

Table 3.5: Recomputation of the right ascension underlying Ptolemy's equation of time

true solar long.	equation of time	error	true solar long.	equation of time	error
6	0;20,33		186	0; 4, 0	
12	0;18, 4		192	0; 2,30	−7
18	0;15,38		198	0; 1,28	+1
24	0;13,24		204	0; 0,37	
30	0;11,13		210	0; 0, 0	
36	0; 9,35		216	0; 0, 2	
42	0; 8,10	+2	222	0; 0,22	
48	0; 7, 7		228	0; 1,13	
54	0; 6,33		234	0; 2,35	
60	0; 6,12		240	0; 4,13	
66	0; 6,34		246	0; 6,34	
72	0; 7, 6		252	0; 9, 6	
78	0; 8, 0		258	0;11,58	
84	0; 9, 7		264	0;15, 2	
90	0;10,23		270	0;18, 9	−1
96	0;11,42		276	0;21,15	
102	0;12,57		282	0;24,11	
108	0;14, 3		288	0;26,50	
114	0;14,53		294	0;29, 4	−1
120	0;15,37		300	0;31, 4	
126	0;16, 3	+20	306	0;32,15	
132	0;16, 2	+23	312	0;33, 6	
138	0;15,18	+5	318	0;33,23	
144	0;14,27	+3	324	0;33, 5	
150	0;13,27		330	0;32,26	
156	0;11,59		336	0;31, 6	
162	0;10,31		342	0;29,30	
168	0; 8,52		348	0;27,33	
174	0; 7,11		354	0;25,21	−1
180	0; 5,33		360	0;23, 0	

Table 3.6: Final recomputation of Ptolemy's table for the equation of time

possibilities for the final digits of the differences.

We can compute an estimate $\hat{c}$ of the epoch constant which minimizes the sum of the squares of the differences between the extracted right ascension and values computed according to method E. If the six outliers are left out, the result is $\hat{c}$ = 3;34,6,26 and an approximate 95 % confidence interval for c is given by ⟨3;34,5,11 , 3;34,7,42⟩. The minimum obtainable standard deviation of the errors in the extracted right ascension values is $3''$, which is equal to the expected standard deviation of a correct extracted table.[110]

3.2.3.6 Final recomputation

In this Section I will give a final recomputation of Ptolemy's table for the equation of time and will make an attempt to explain some of the outliers. If we assume that Ptolemy had accurate solar equation values, that he determined the right ascension according to method E in Section 3.2.3.5, and that he used the same method of rounding that we use today,[111] the choice c = 3;34,7,30 for the epoch constant leads to the minimum possible number of errors in the table for the equation of time, namely 10. Of these errors seven may be called true outliers: they will not disappear for any value of c in the range 3;34,0 to 3;34,15.[112] These true outliers occur for arguments 42, 126, 132, 138, 144, 192 and 198, and caused the errors that we found in the extracted right ascension and solar equation. Since the arguments of the outliers are not symmetrical, we conclude that the outliers do not derive from errors in the underlying tables.

Note that the table for the equation of time also has three small errors for arguments for which the corresponding extracted solar equation and right ascension values are correct. The reason this is possible is that the maximum differences of $7\frac{1}{2}''$ between the extracted and recomputed solar equation and right ascension values (see Tables 3.3 and 3.5) may lead to maximum differences of $\frac{1}{15}(7\frac{1}{2}'' + 7\frac{1}{2}'') = 1''$ between the equation of time values and their recomputation.

The errors in the final recomputation for epoch constant c = 3;34,7,30 are displayed in Table 3.6, where the first and fourth columns contain the true solar longitude, the second and fifth, Ptolemy's table for the equation of time, and the third and sixth, the errors.

I have checked the possibility that the errors in Ptolemy's table for the equation of time result from scribal errors in the Greek transmission of the table by inspecting photographs of five other manuscripts of the *Handy Tables*, namely Vatican gr. 1291, Laurentianus gr. 28/26 and Laurentianus gr. 28/48 (Florence), Marcianus gr. 325 (Venice) and Ambrosianus gr. H 57 sup (Milan).[113] It turned out that in four of these five manuscripts

[110] It can be checked that for the value $\hat{c}$ of the epoch constant the errors in the extracted right ascension values all have the same order of magnitude and show no obvious dependency. Thus the conditions for the least squares estimation are satisfied.

[111] See my remarks in Section 3.2.2.3.

[112] We need not consider values of c outside this interval, since for such values the total number of errors in the table amounts to at least 25.

[113] Descriptions of three of these manuscripts can be found in Tihon 1978, pp. 139–141 (Laurentianus gr. 28/26), pp. 103–104 (Laurentianus gr. 28/48) and pp. 88–90 (Ambrosianus gr. H 57 sup). Furthermore,

the tabular value for true longitude 42 is 0;8,8, equal to the recomputed value. The same four manuscripts have $T(138)$ = 0;15,19 instead of 0;15,18, but here the recomputed value is 0;15,13. Two manuscripts have $T(192)$ = 0;2,31 instead of 0;2,30, whereas the recomputed value is 0;2,37. Two other manuscripts display $T(198)$ = 0;1,26 instead of 0;1,28, whereas the recomputed value is 0;1,27.[114]

We conclude that the error for argument 42 can be explained from an error in the transmission of the table. Furthermore the error for argument 192 could easily result from a scribal mistake (Β Λ instead of Β ΛΖ). I have not been able to find an explanation for the errors for arguments 126 to 144, which seem to be correlated. Note that these errors must have been made before the interpolation was performed, since the interpolation pattern is regular.[115]

3.2.4 Conclusions

The mathematical analysis in the previous section has enabled us to recover practically every detail of the method that Ptolemy used for the computation of his table for the equation of time. From primary sources and from work by Rome and Neugebauer[116] we know the following:

- The independent variable of Ptolemy's table for the equation of time is the true solar longitude.
- The epoch constant c was chosen in such a way that all tabular values were non-negative and had to be added to true solar time to obtain mean solar time.

From this information we concluded that Ptolemy must have calculated his table for the equation of time according to the general formula $E_h(\lambda) = \frac{1}{D}\left(\alpha(\lambda) - \lambda + q(\lambda) + c\right)$ (where D is the conversion factor, $\alpha(\lambda)$ the right ascension of the true sun and $q(\lambda)$ the solar equation as a function of the true solar position). Our analysis confirmed that the table has the true solar longitude for the independent variable and furthermore led to the following new results:

- The conversion factor D is equal to 15.
- The underlying solar equation $q(\lambda)$ is based on the value 2;30 for the eccentricity, not on 2;29,30. Instead of the value 65;30 for the solar apogee, which is mentioned in all historical sources concerning the *Almagest* or the *Handy Tables*, Ptolemy used the

a table of contents of Laurentianus gr. 28/26 is presented in Tihon 1992, pp. 64–66. The manuscript Vatican gr. 1291 is described in Tihon 1992, pp. 61–64. A table of contents of this manuscript can also be found in Neugebauer 1975, vol. 2, pp. 977–978.

[114]I have not checked the possibility that scribal errors in internodal values are the cause of irregularities in the interpolation pattern.

[115]The only large irregularity in the interpolation pattern occurs between arguments 126 and 132, where the tabular differences are $+6, +6, +6, -6, -6, -7''$. In the table the local maximum in this interval occurs for argument 129 and amounts to 0;16,21. If we use Ptolemy's method of computation for the nodes to calculate equation of time values for every integer argument, we find a maximum 0;15,49 for true solar longitude 130, completely different from the tabular value.

[116]See footnote 52.

rounded value 66;0. The underlying solar equation was not determined by means of inverse linear interpolation in a table for the solar equation as a function of the mean solar longitude, as suggested by Theon of Alexandria, but was computed independently and was accurate to at least seconds.[117]

- The right ascension $\alpha(\lambda)$ underlying Ptolemy's table for the equation of time is based on the values for multiples of 10° given in the *Almagest*. The intermediate values were determined by means of exact linear interpolation without rounding. The underlying value of the obliquity is 23;51,20.[118]
- Neither numerical nor historical considerations make it possible to decide on a definite value for the epoch constant. The number of errors in both the extracted right ascension table and our final recomputation of the equation of time table is minimized for $c = 3;34,7,30$. A 95 % confidence interval for c based on all tabular values for multiples of 6° excluding the seven outliers that we have found is $\langle 3;34,4,45\,,\ 3;34,7,4\,\rangle$.
- Linear interpolation was used to determine the tabular values for non-multiples of 6°. The tabular differences seem to be distributed irregularly over the internodal values.

From these results we see that in computing his table for the equation of time Ptolemy was a very practical astronomer. He simplified the calculations in three different ways: by rounding the apogee value in order to avoid interpolation between his solar equation values, by using linear interpolation between right ascension values for every 10°, and by using some type of linear interpolation between equation of time values for multiples of 6°. We can assume that Ptolemy was aware that the resulting errors were small (it can be checked that the maximum difference between Ptolemy's equation of time values and precisely calculated values is at most 15 seconds of time). From our results it also becomes clear that one must be very careful when comparing Ptolemy's tables with recomputations based on modern formulae. Incorrect conclusions concerning the underlying parameters or the dependence on other tables may easily be drawn if one ignores the possibility that Ptolemy made use of inexact methods that simplified the computations.

We have noted that in his *Great Commentary* Theon gives correct descriptions of the mathematical structure of the declination table and the right ascension table in Ptolemy's *Handy Tables*. He also correctly describes the interpolation pattern in the solar equation table, but fails to mention that the nodal values are not all equal to the corresponding values in the *Almagest*.

Our analysis reveals that there are several divergences between Theon's description in the *Great Commentary*[119] and the actual computation of Ptolemy's table for the equation of time. Firstly, Theon does not mention the interpolation within intervals of 6 degrees and even gives his numerical example for $\lambda = 271$, which is not a node. Secondly, he uses the right ascension which occurs in the column adjacent to the equation of time instead of exact interpolation within intervals of 10 degrees. Thirdly, he uses inverse interpolation in the table for the solar equation as a function of the mean anomaly instead of accurate

[117] See Section 3.2.3.4.
[118] See Section 3.2.3.5.
[119] Mogenet & Tihon 1985, pp. 99–100 and 163–164.

values from a table for the solar equation as a function of the true anomaly. It seems almost certain that Theon did not know precisely how Ptolemy's table for the equation of time had been computed.

Although we have shown that the table for the equation of time in the *Handy Tables* was computed differently from what we expected on the basis of available historical information and other tables from the *Almagest* and the *Handy Tables*, we can conclude that there is no reason to question Ptolemy's authorship of the table. We have seen that the particular type of linear interpolation which was used in the right ascension and various other tables in the *Handy Tables* was not used in the right ascension underlying the table for the equation of time, or in the equation of time table itself. Therefore it seems possible that Ptolemy already had accurate equation of time values before he compiled the *Handy Tables* and for some reason left these unchanged when he modified many of the other tables that he included in the *Handy Tables*.

3.2.5 The table for the equation of time in the papyrus London 1278

The Greek papyrus London 1278, kept in the British Museum, contains six fragments of numerical tables. Neugebauer made a careful analysis of the papyrus[120] and suggested that the tables it contains may be recensions of tables found in Ptolemy's *Handy Tables*.[121] Neugebauer concluded that the fragments 6^v, 3^r, 3^v, 1^r and 2^r in that order constitute a badly damaged trigonometric table in the Ptolemaic tradition, displaying the sine of the right ascension, the right ascension itself, and the equation of time for every degree of the ecliptic, starting with Aries.[122] It turns out that the right ascension values are identical to those in the *Handy Tables* apart from a difference of 90° resulting from the fact that the *Handy Tables* tabulate the *normed* instead of the ordinary right ascension.[123] The values for the sine of the right ascension can also be found in the *Handy Tables* in connection with the determination of the length of daylight.[124] The equation of time values, however, are different from those in the *Handy Tables* analysed in Section 3.2.3. From the relative positions of the extremes, it can be seen that the equation of time in P. London 1278 was computed for the Era Nabonassar. This was the era utilized in the *Almagest*, whereas the *Handy Tables* make use of the Era Philip.[125] The minimum equation of time in P. London 1278 occurs in Aquarius and the equation must always be subtracted from true solar time to obtain mean solar time. The equation of time values are given to an accuracy of minutes of an hour, whereas the *Handy Tables* give values to seconds. Because of the epochs involved Neugebauer suggested that the equation of time table in P. London 1278 is a version intermediate between the *Almagest* and the *Handy*

[120] Neugebauer 1958.
[121] Neugebauer 1958, pp. 109–112.
[122] Neugebauer 1958, pp. 97–103.
[123] See Section 1.3 for more information about the normed right ascension; see Section 3.2.2.2 on the right ascension table in the *Handy Tables*.
[124] See Neugebauer 1958, pp. 102–103 and Stahlman 1959, pp. 265–266.
[125] Cf. Section 3.1.1 (p. 100) of this thesis and Neugebauer 1958, pp. 109–111.

Tables. I will now further investigate this matter by analysing the equation of time values found in the papyrus.

Since the papyrus London 1278 contains only 72 equation of time values to minutes, occurring in scattered groups, it would normally be difficult to determine the underlying parameter values and mathematical structure. However, by means of a least squares estimation and by comparing the separate tabular values with those in the *Handy Tables*, it will be possible to draw conclusions about both. I will show that there is reasonable evidence to assume that the equation of time found in the papyrus in the British Museum is directly related to the equation of time in the *Handy Tables*.

Table 3.7 displays the 72 legible equation of time values in P. London 1278. To obtain a first impression of the underlying parameters and the mathematical structure of the table, I applied a least squares estimation using all available values. Assuming that the conversion factor D equals 15 and that the independent variable of the table is the true solar longitude, we find a minimum possible standard deviation of $20''$ and the following approximate 95 % confidence intervals for the underlying parameters:[126]

parameter	*95 % confidence interval*
obliquity of the ecliptic ε	$\langle$ 23;20,22 , 23;38, 2 $\rangle$
solar eccentricity e	$\langle$ 2;26,34 , 2;32,13 $\rangle$
solar apogee λ_A	$\langle$ 65;44,48 , 66;53,54 $\rangle$
epoch constant c	$\langle$ 4;26, 6 , 4;28,48 $\rangle$

Since the tabular errors of a correct table with values to minutes are expected to have a standard deviation of approximately $17''$, we can conclude that the equation of time table in P. London 1278 fits into the Ptolemaic tradition and was probably computed according to formula (3.5). For two reasons I consider it less probable that the table was computed according to formula (3.6), i.e. that the independent variable of the table is the mean solar longitude. Firstly, even though the minimum obtainable standard deviation of the errors would be even smaller (namely $18''$), the approximate 95 % confidence intervals would no longer contain the historically plausible values 2;29,30 or 2;30 for the eccentricity and 65;30 or 66;0 for the apogee, or any other plausible values. Secondly, we have seen that the equation of time table in the *Handy Tables* had the true solar longitude as its independent variable, and that none of the relevant sources related to Ptolemy suggests the possibility that the mean solar longitude might be the independent variable of a table for the equation of time.[127] Consequently, I will assume in the sequel that the independent variable of the equation of time table in P. London 1278 is the true solar longitude.

The approximate 95 % confidence intervals suggest that, as in the case of the *Handy Tables*, the underlying value for the eccentricity is $e = 2;30$ (or possibly $e = 2;29,30$), the value for the apogee $\lambda_A = 66°$. The confidence interval for the obliquity contains only values much smaller than the attested value $\varepsilon = 23;51,20$. Since the same phenomenon

[126]The results of the least squares estimation are essentially the same if we take $D = 15;2,28$. Since there is no reason so far to believe that $D = 15;2,28$ was used for the computation of tables for the equation of time by Greek astronomers, I will take D equal to 15 in the remainder of this section.

[127]See Sections 3.2.2.2 and 3.2.3.2.

λ	Aries	Taurus	Gemini	Cancer	Leo	Virgo
1				0;21		
2				0;21		
3				0;21		
4				0;21		
5				0;21		
6			0;25	0;20		
7		0;23	0;25	0;20		
8		0;23	0;25	0;20		
9		0;23		0;20		
10		0;23		0;20		
11		0;24		0;19		
12		0;24		0;19		
13		0;24		0;19		
14		0;24				
15		0;24				
16						
λ	Libra	Scorpio	Sagittarius	Capricornus	Aquarius	Pisces
1		0;32	0;28	0;13		
2		0;32	0;27	0;13		
3		0;32	0;27	0;12		
4		0;32	0;26	0;12		
5		0;32	0;26	0;11		
6		0;32	0;25	0;11	0; 0	
7		0;32	0;25	0;10	0; 0	
8		0;32	0;24	0;10	0; 0	
9		0;32	0;24	0; 9	0; 0	
10		0;32	0;24	0; 9	0; 0	
11		0;32	0;23	0; 8	0; 0	
12		0;32	0;23	0; 8	0; 0	
13		0;32	0;22			
14			0;22			
15			0;21			
16						

Table 3.7: The equation of time values in the papyrus London 1278

occurs for the equation of time in the *Handy Tables*,[128] it seems possible that the right ascension used for the calculation of the equation of time values in P. London 1278 contains errors similar to those in the right ascension underlying the equation of time in the *Handy Tables* or even that the equation of time table in P. London 1278 is directly related to the one in the *Handy Tables*.

In the remainder of this Section I will investigate the possibility that the equation of time values in P. London 1278 were computed by subtracting the values in the *Handy Tables* from a constant and rounding the results to minutes, i.e.

$$T_{PL}(\lambda) = r_1(C - T_{HT}(\lambda)) \tag{3.12}$$

for every λ, where C is a constant, $T_{PL}(\lambda)$ are the values in P. London 1278, $T_{HT}(\lambda)$ those in the *Handy Tables*, and r_1 indicates (modern) rounding to minutes. Thus I will consider the distribution of $\Sigma(\lambda) \stackrel{\text{def}}{=} T_{PL}(\lambda) + T_{HT}(\lambda)$ for all λ for which P. London 1278 displays values. It turns out that $\Sigma(\lambda)$ lies in the range from 0;31,34 to 0;32,24 for 62 values of λ, and that furthermore

$$\begin{aligned} \Sigma(100) &= 0;32,32, & \Sigma(308) &= 0;32,32,\\ \Sigma(223) &= 0;32,30, & \Sigma(309) &= 0;32,41,\\ \Sigma(241) &= 0;32,36, & \Sigma(310) &= 0;32,49,\\ \Sigma(248) &= 0;31,24, & \Sigma(311) &= 0;32,58,\\ \Sigma(255) &= 0;31,31, \quad \text{and} & \Sigma(312) &= 0;33,\ 6. \end{aligned}$$

We note that for C = 0;32 (corresponding to a difference of 8;0 between the epoch constants of both tables), 64 out of the 72 available equation of time values in P. London 1278 were correctly computed according to formula (3.12) if the rounding was performed in the modern way.[129] The values $\Sigma(\lambda)$ outside the range 0;31,30 to 0;32,30 for $\lambda = 100, 241$ and 248 could be attributed to small rounding errors. The values $\Sigma(\lambda)$ for $\lambda = 308, 309, \ldots, 312$ are clearly correlated and could be explained by assuming that the author of the table in P. London 1278 set $T_{PL}(\lambda)$ to 0;0 whenever $r_1(0;32 - T_{HT}) < 0$.

Conclusions. *The equation of time table in the papyrus London 1278 was computed in accordance with the theory developed by Ptolemy in the Almagest. It seems probable that the table was directly computed from the equation of time in the Handy Tables using formula (3.12), where C was taken equal to* 0;32*, the rounding was performed in the modern way, and the resulting negative values in the sign Aquarius were set to zero. Thus, like the equation of time in the Handy Tables, the table in P. London 1278 is based on parameter values* ε = 23;51,20*,* e = 2;30 *and* λ_A = 66;0*.*

A possible explanation why the author of P. London 1278 used the value 0;32 for C, whereas the maximum of the equation of time in the *Handy Tables* is 0;33,23, is as

[128] I performed least squares estimations on the equation of time values in the *Handy Tables* in Section 3.2.3.2, but did not give all the results. If all tabular values are used, the least squares estimate for the obliquity is $\hat{\varepsilon}$ = 23;44,59. If only those arguments are used for which P. London 1278 has tabular values as well, the estimate is $\hat{\varepsilon}$ = 23;41,50. This is very close to the estimate $\hat{\varepsilon}$ = 23;41,44 obtained from the table in the papyrus if we leave out the zero values in the sign Aquarius. I will explain below that those values may have been set to zero to avoid negative equation of time values.

[129] The same would hold for C = 0;32,30 if truncation were used.

follows. In Section 3.1.1 it was noted that Ptolemy assumed that mean and true solar time at epoch were equal. As Theon explained in his *Great Commentary on the Handy Tables*, Ptolemy made the minimum equation of time in his table equal to zero in order to obtain an equation that must always be added to true solar time.[130] Thus he neglected the fact that the solar position at the epoch 1 Philip of the *Handy Tables* was 17° Scorpio, whereas the minimum equation of time occurred for 0° Scorpio. Consequently, for every true solar position, the actual equation of time was approximately $1'4''$ smaller than the value given in the *Handy Tables*.[131] For the Era Nabonassar the same problem occured: the true solar position at epoch was 3° Pisces, whereas the minimum equation of time was assumed for 18° Aquarius. To tabulate the equation of time correctly, the epoch constant must be chosen in such a way that the equation for 3° Pisces became zero. This implies that, when computing a table for the Era Nabonassar from the equation of time in the *Handy Tables* by means of formula (3.12) above, C should be taken equal to the tabular value for $\lambda = 333$, i.e. to 0;31,47, which is rounded to $C =$ 0;32. Instead of accepting the negative values that would thus arise for longitudes 308 to 329, the author of the equation of time table in P. London 1278 apparently preferred to make all these values equal to zero.

Thus we see that the equation of time tables in the *Handy Tables* and in P. London 1278 present two different solutions to the problem that the actual maximum or minimum equation does not occur precisely at epoch. In the case of the *Handy Tables* a small constant was added to all tabular values in order to make them non-negative; in the case of P. London 1278 all negative values were simply set to zero.

Note that the relationship between the equation of time tables in the *Handy Tables* and in P. London 1278 need not necessarily be the one described above. Another possibility is that both were computed from a non-extant equation of time table to seconds for the Era Nabonassar. The errors that we found in the *Handy Tables*[132] cannot be used to obtain more detailed information concerning this matter, since either they are too small or they occur in regions where P. London 1278 has no tabular values at all.

3.3 Kushyār ibn Labbān

3.3.1 Historical Context

Kushyār ibn Labbān ibn Bāshahrī Abu'l-Ḥasan al-Jīlī flourished around the year 1000 in Baghdad.[133] Kushyār's main achievements were in the fields of arithmetic, trigonometry and astronomy. He wrote a work "The Elements of Hindu Reckoning" about sexagesimal arithmetic and computed extensive trigonometric tables. In his astronomical works Kushyār made use of the parameters of al-Battānī (fl. around 900) instead of making his

[130] See the information on the epoch constant in Section 3.2.2.2 and Mogenet & Tihon 1985, pp. 123, lines 9–14; French translation on p. 182.

[131] This difference can be found in the equation of time table as the value for argument 17° Scorpio.

[132] See Section 3.2.3.6.

[133] More information about Kushyār ibn Labbān can be found in Section 4.1.2 of this thesis or in the article by A.S. Saidan in the Dictionary of Scientific Biography (*DSB*).

own observations. As we will see below, he made his planetary tables easier to use by applying so-called "displaced equations".

It is unclear whether Kushyār wrote one or two astronomical handbooks. In "The Book of the Astrolabe" he mentions the *Jāmiʿ Zīj* ("Comprehensive Astronomical Tables") and the *Bāligh Zīj* ("Extensive Astronomical Tables") as two different works. Since the four manuscripts of Kushyār's zīj(es) that I consulted (Istanbul Fatih 3418, Berlin Ahlwardt 5751, Leiden Or. 8 (1054), and Cairo DM 188/2) basically contain the same set of tables, I will refer to each of these manuscripts as the *Jāmiʿ Zīj*.[134]

3.3.2 Preliminaries

In the manuscript Istanbul Fatih 3418 a table for the equation of time can be found on folio 46ʳ. This table displays minutes and seconds of the equation of time for every 6 degrees of mean solar longitude, along with interpolation coefficients (ḥiṣṣa al-daraja) in seconds and thirds.[135] The heading of the table states that the table must be entered with the mean solar longitude and that the equation of time thus found must always be subtracted from the previously found (true solar) time. The solar apogee is explicitly mentioned to be in 24° Gemini. Alongside the table we find instructions for the use of the interpolation coefficients.

On folios 7ʳ–7ᵛ of the Istanbul manuscript we find in Section 5 of Chapter 4 of the First Treatise the following instructions for determining the equation of time:[136]

> ON THE EQUATION OF TIME
> For the time of the exact positions of sun and moon, there is a correction which is known as the equation of time.
> If we want this [correction], we subtract from the mean solar position at the time ten signs and sixteen degrees, and what remains is the "outcome of the mean". And we subtract from the right ascension of the true solar position at the time ten signs and twenty two degrees and four minutes, and what remains is the "outcome of the ascension". Next we take the excess of the outcome of the mean over the outcome of the ascension and multiply it by four. Next we make the sign minutes and the minutes seconds, and we have minutes of an hour of the equation of time. We subtract them from the time corrected for the longitude difference and we have the time corrected for the equation of time. (...)[137]
> And according to this arithmetical procedure we set up a table in which we wrote the mean solar longitude and opposite it minutes and seconds of an hour of the equation of time, so that it is not necessary to calculate the sun twice, on condition that the apogee is in twenty-four of Gemini. And as far as the motion of the apogee is concerned, there is no perceivable effect on this equation except in very long periods. And for the determination of the exact positions of the five planets there is definitely no need for this equation.

[134] More information about the manuscripts and about the tables that they contain can be found in Section 4.1.2 of this thesis.

[135] The interpolation coefficients were computed by dividing the differences between consecutive tabular values by 6.

[136] My own translation. Text between square brackets has been added.

[137] The following six lines explain "another method", which is equivalent to the one above.

mean solar motion	equation of time	error	mean solar motion	equation of time	error
0	0; 9,48	+4	180	0;24, 8	−4
6	0;11,52		186	0;26, 4	
12	0;13,52	−4	192	0;27,44	
18	0;15,52	−4	198	0;29,16	
24	0;17,40		204	0;30,24	
30	0;19, 8	−4	210	0;31,12	
36	0;20,16	−8	216	0;31,32	
42	0;21,12	−4	222	0;31,24	
48	0;21,48		228	0;30,48	
54	0;21,56		234	0;29,40	+4
60	0;21,44		240	0;27,56	
66	0;21, 8	−4	246	0;25,48	
72	0;20,20		252	0;23,16	
78	0;19,16		258	0;20,32	+4
84	0;17,56	−4	264	0;17,24	
90	0;16,44		270	0;14,20	+4
96	0;15,24		276	0;11,12	
102	0;14, 8	−4	282	0; 8,20	
108	0;13,16	+4	288	0; 5,48	
114	0;12,28		294	0; 3,40	+4
120	0;12, 0	−4	300	0; 1,56	
126	0;11,56		306	0; 0,44	
132	0;12,12	−4	312	0; 0, 8	+4
138	0;12,52		318	0; 0, 0	+4
144	0;13,52	+4	324	0; 0,20	
150	0;15, 8		330	0; 1, 8	
156	0;16,36	−4	336	0; 2,24	+4
162	0;18,24		342	0; 3,52	
168	0;20,12	−4	348	0; 5,40	
174	0;22,12		354	0; 7,40	

Table 3.8: Kushyār's table for the equation of time (Istanbul Fatih 3418, folio 46^{r})

In the other three manuscripts mentioned above we find the same table for the equation of time. However, in these manuscripts no interpolation coefficients are given; instead the intermediate tabular values for every single degree of mean solar longitude have been filled in. The resulting interpolation pattern is extremely regular. The same holds for another copy of Kushyār's table for the equation of time which is found on folio 62^{v} of Escorial Ms. árabe 927, a recension of the Mumtaḥan Zīj.[138] In this copy, however, the column for Sagittarius has been badly damaged by scribal errors. The instructions for determining the equation of time translated above are also present in the Leiden[139] and Escorial[140] manuscripts, but are missing from the Berlin manuscript.[141]

3.3.3 Recomputation

Table 3.8 displays the equation of time values for every 6 degrees of mean solar longitude as found on folio 46^{r} of Istanbul Fatih 3418 (the "error" column will be explained below). Inspection of tabular differences and the tabulated interpolation coefficients led to the correction of a single scribal error: $T(84) = 0;17,16$ must be corrected to 0;17;56. I will not consider the intermediate tabular values as given in the other manuscripts, since it is clear that they were determined using the interpolation coefficients of the Istanbul table.

We first note that all tabular values have a number of seconds which is a multiple of four. We conclude that Kushyār used conversion factor $D = 15$ and that he calculated the equation of time with an accuracy of equatorial minutes.

Next we consider the right ascension as tabulated by Kushyār. There is an obvious difference between the tables in the Istanbul and Cairo manuscripts on the one hand[142] and those in the Berlin and Leiden manuscripts on the other.[143] The right ascension table in Berlin and Leiden occupies two pages, was calculated for obliquity 23;35, and contains only eight errors. The right ascension table in the Istanbul and Cairo manuscripts is compressed to a single page. It contains a large number of errors, which are due to the use of linear interpolation within intervals of 6 degrees. Although 4 of the 15 nodes are in error as well, there is no doubt that the underlying obliquity value is 23;35.[144]

As far as the solar equation is concerned, Kushyār makes use of what I will call a "displaced equation": to avoid a partially additive and partially subtractive solar equation, he instead tabulates $2 - \bar{q}(\bar{\lambda})$ (with $\bar{q}(\bar{\lambda})$ as in formula 3.4). To obtain the true solar position

[138]See Kennedy 1956a, p. 132 , no. 51 and pp. 145–147 for information about the Mumtaḥan Zīj. The table for the equation of time concerned occurs on p. 121 of the facsimile edition Yaḥyā ibn Abī Manṣūr.

[139]Leiden Or. 8 (1054), folios 8^{r}–8^{v} (Section 26 of the First Treatise). There are various small differences between the instructions in the Leiden manuscript and those in the Istanbul manuscript. For instance, the "second method" is missing and the number to be subtracted from the right ascension of the true solar position has *eight* minutes instead of four.

[140]Escorial Ms. árabe 927, folio 63^{r}, or Yaḥyā ibn Abī Manṣūr, p. 122. This text is very close to the one in the Istanbul manuscript.

[141]Cf. Ahlwardt 1893, pp. 203–206. In the table of contents on pages 2–4 a section on the equation of time is included as Section 20 of the First Treatise, but the text omits Sections 9 to 69.

[142]Istanbul Fatih 3418, folio 84^{v}; Cairo DM 188/2, folio 12^{v}.

[143]Berlin Ahlwardt 5751, pp. 148–149; Leiden Or. 8, folios 76^{v}–77^{r}.

[144]See the analysis of al-Baghdādī's oblique ascension table for latitude 36° in Section 4.3.13.2 of this thesis.

the displaced equation must always be added to the displaced mean longitude $\widetilde{\lambda} \stackrel{\text{def}}{=} \bar{\lambda} - 2$. Kushyār calls $\widetilde{\lambda}$ the mean solar longitude. Consequently his table for the solar equation displays $\widetilde{q}(\widetilde{a}) \stackrel{\text{def}}{=} 2 - \bar{q}(\bar{a}) = 2 - \bar{q}(\widetilde{a} + 2)$, where $\bar{a} = \bar{\lambda} - \lambda_A$ is the mean solar anomaly and the argument of the table, $\widetilde{a} = \widetilde{\lambda} - \lambda_A$, the displaced mean solar anomaly.

Kushyār's solar equation table was badly computed.[145] Like al-Battānī's solar equation table it was based on eccentricity $e = 2;4,45$ and has an irregular error pattern that may at least partially be due to interpolation. The error patterns in al-Battānī's and Kushyār's tables are very different between arguments 90 and 135, but almost identical outside this interval. Neither in Kushyār's Jāmiʿ Zīj,[146] nor in al-Battānī's Ṣābiʾ Zīj[147] have I been able to find clues as to the possible causes of the errors.

Now we turn to the equation of time, which is given as a function of "the mean solar longitude", i.e. presumably of the displaced mean solar longitude $\widetilde{\lambda}$. Denoting the displacement by Δ (Δ may be both positive and negative) and the displaced equation of time by $\widetilde{E}_h$, we have in general[148]

$$\widetilde{E}_h(\widetilde{\lambda}) = \bar{E}_h(\widetilde{\lambda} + \Delta) = \frac{1}{d}\left(\widetilde{\lambda} + \Delta - \alpha(\widetilde{\lambda} + \Delta - \bar{q}(\widetilde{\lambda} + \Delta)) + c\right). \qquad (3.13)$$

We have already noted that for Kushyār's table the conversion factor $D = 15$ and the displacement $\Delta = 2$. Relying on the text accompanying the table for the equation of time we expect $\lambda_A = 84°$ for the solar apogee and $c = 4;4$ for the epoch constant (N.B. the difference between $10^s22°4'$ and $10^s16°$ mentioned in Kushyār's instructions equals $\Delta + c$). Finally we know both from tables and text by Kushyār and al-Battānī that they used obliquity of the ecliptic $\varepsilon = 23;35$ and solar eccentricity $e = 2;4,45$.

The "error" column of Table 3.8 displays the differences between Kushyār's table for the equation of time and a recomputation using formula (3.13) and the above-mentioned parameter values. Note that I rounded the equation to minutes of arc as did Kushyār. Of the 60 tabular values 23 show an error; the standard deviation of the errors is $2''38'''$.[149] We note that there is a definite improvement compared to a recomputation by Kennedy.[150] Nevertheless, the remaining errors in my recomputation seem to be dependent: they are mainly negative in the first two quadrants and exclusively positive in the third and fourth quadrants. In fact the residual plot in Figure 3.3 shows an upward trend.

From a least squares estimation it appears that no significantly better agreement with Kushyār's table can be achieved by varying the parameter values. The minimum obtainable standard deviation of the errors is $2''15'''$, and 95 % confidence intervals for the underlying parameters are given by:

145 The table can be found in Istanbul Fatih 3418, folios 46^v–48^r; Cairo DM 188/2, folios 17^v–18^v; Berlin Ahlwardt 5751, pp. 70–73; and Leiden Or. 8 (1054), folios 38^r–38^v.

146 See Istanbul Fatih 3418, folios 7^v, 105^v–106^r and 143^r–143^v.

147 See Nallino 1899–1907, vol. 3, pp. 64–73 (Arabic) or vol. 1, pp. 43–48 (Latin translation).

148 Cf. formula (3.6).

149 If the tabular values were precisely computed, the standard deviation of the errors would be approximately $1''9'''$, namely $4 \cdot \frac{1}{\sqrt{12}} 60^{-2}$; cf. Section 1.2.4.

150 Kennedy 1988, pp. 2–4. Kennedy did not take into account that Kushyār used a displaced mean motion and found errors with a standard deviation of $32''$.

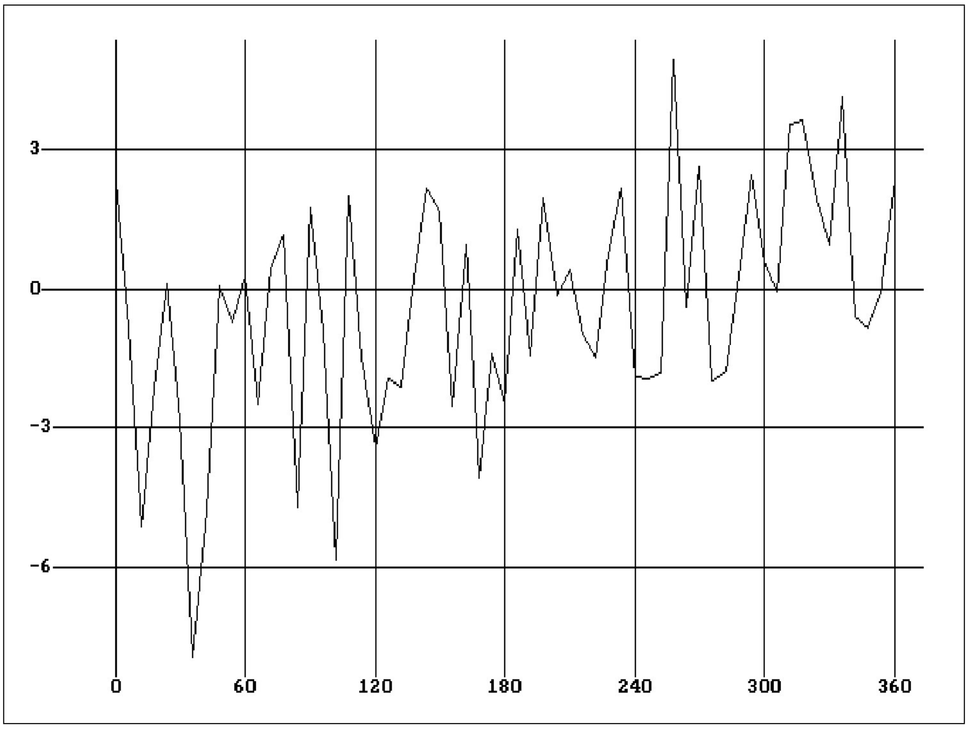

Figure 3.3: Errors (in seconds) in Kushyar's table for the equation of time

parameter	*95 % confidence interval*
obliquity of the ecliptic	⟨ 23;33, 4 , 23;35, 0 ⟩
solar eccentricity	⟨ 2; 4,27 , 2; 4,55 ⟩
solar apogee	⟨ 84; 1,48 , 84;14,26 ⟩
epoch constant	⟨ 4; 3,44 , 4; 4, 2 ⟩
displacement	⟨ 1;57,37 , 2; 2,27 ⟩

It turns out that a shift of the parameters to (unattested) values closer to the middle of these confidence intervals leads to a somewhat *larger* number of errors in the recomputation. Furthermore there are no historically plausible values for either of the parameters which would improve on our recomputation. I tried various other possibilities to explain the upward trend in the residuals, e.g. the use of Kushyār's solar equation values instead of accurately computed ones and rounding of the true solar position before the right ascension is computed. None of these possibilities led to a better recomputation.

3.3.4 Conclusions

We conclude that Kushyār computed the table for the equation of time in the Second Treatise of his *Jāmiʿ Zīj* according to the rules presented in the First Treatise. The argument of the table is the displaced mean solar longitude, which Kushyār introduced to make his solar equation always additive. The table was computed with an accuracy of equatorial minutes; the use of the conversion factor $D = 15$ yielded tabular values which are all multiples of 4″. The underlying obliquity $\varepsilon = 23;35$ and solar eccentricity $e = 2;4,45$ are the values used consistently by both al-Battānī and Kushyār. I will comment in more detail on the values for the solar apogee and the epoch constant.

Kushyār's table for the equation of time was computed using the value $\lambda_A = 84°$ for the solar apogee, in agreement with the information in the heading of the table. We will see that a more accurate value for the apogee is mentioned in the text, from which we conclude that 84° is only an approximate value for the year in which Kushyār compiled the Jāmiʿ Zīj.

On folio 6ᵛ of Istanbul Fatih 3418, Kushyār describes the determination of the apogees of the planets. He states that in the year 1 Yazdigird the solar apogee was located in 78°31′ and that its motion was 54″ per Persian year of 365 days, i.e. 1° per $66\frac{2}{3}$ Persian years. On folio 45ᵛ we find a table for the motion of the apogees based on the same parameter. Furthermore three lists of apogee values are given for the years 1, 249 and 331 Yazdigird. In these lists the values for the solar apogee are 78°31′, 82°14′ and 83°28′ respectively. The first and second lists are attributed to al-Battānī. Indeed, 249 Yazdigird corresponds to the Seleucid year 1191 (880 A.D.), the epoch of al-Battānī's tables, and the apogee value given for this year is in agreement with al-Battānī's value 82°17′ which he determined in the Seleucid year 1194.[151] Kushyār's motion of the apogee does not correspond completely to al-Battānī, who states that the apogee moves 1° in 66 Seleucid years or in 68 Hijra years.[152] From the third list we can conclude that Kushyār calculated his tables in or shortly after 331 Yazdigird (962 A.D.). The apogee value 84° (corresponding to the year 997) found in the table for the equation of time is apparently a rounded value, chosen in a such a way that the table would be accurate for at least the coming 70 years.

We have seen that Kushyār computed his table for the equation of time with epoch constant $c = 4;4$. Thanks to the explanatory text it is easy to discover the origin of this value. In fact the $10^s16°$ mentioned there is close to the value of the displaced mean solar position for which the equation of time assumes its minimum value. Since $10^s22°4'$ is approximately equal to the right ascension of the true solar position $10^s19°38'21''$ (which is obtained by adding the displaced solar equation 3;38,21 to $10^s16°$), it follows that the equation of time thus obtained is always non-negative and must always be subtracted from true solar time to obtain mean solar time.[153] Note that the mean solar position at the epoch 1 Yazdigird of Kushyār's mean motion tables was approximately 84°55′,[154] the equation of time approximately 17 minutes of time. It is not yet clear to me whether Kushyār neglected these minutes or compensated for them in the epoch value of the mean solar motion.[155]

Contrary to earlier authors[156] Kushyār tabulated the equation of time as a function of the mean solar longitude. Thus he did not need to compute the solar equation as a

[151] Nallino 1899–1907, vol. 3, pp. 64–67 (Arabic) or vol. 1, pp. 43–44 (Latin translation).

[152] See Nallino 1899–1907, vol. 3, p. 108 (Arabic) or vol. 1, p. 72 (Latin translation). The average length of the Seleucid year is $365\frac{1}{4}$ days, of the Arabic year $354\frac{11}{30}$ days.

[153] The fact that my recomputation shows a negative value for displaced mean solar longitude 318 may be of help in the determination of Kushyār's precise method of computation.

[154] See the table for solar mean motion on folios 44ᵛ–45ʳ of Fatih 3418.

[155] Cf. Neugebauer 1962, pp. 63–65.

[156] In particular Ptolemy, al-Khwārizmī and al-Battānī.

function of the true solar longitude, but instead had to carry out an interpolation in a right ascension table. As was explained in Section 3.1.1, the practical difference between using a table for the equation of time with the mean solar longitude as its independent variable and one with the true solar longitude is small.

Since the tables for the equation of time in the four manuscripts of Kushyār's zīj(es) that I inspected are identical apart from the interpolation coefficients in Istanbul Fatih 3418, we cannot draw any further conclusions about the number of zījes that Kushyār compiled. As I noted in Section 3.3.1, the differences between the four sets of tables are generally small.

3.4 The Baghdādī Zīj

3.4.1 Historical Context

The manuscript Paris Bibliothèque Nationale Arabe 2486 is a unique copy of a zīj completed in the year 1285 by a certain Jamāl al-Dīn Abī al-Qāsim ibn Maḥfūz al-munajjim al-Baghdādī. In Chapter 4 of this thesis the reader can find information about the author and a more detailed description of the manuscript (Section 4.1), as well as extensive analyses of the trigonometric and spherical astronomical tables that occur in the zīj (Section 4.3). These analyses extend the conclusions of other authors[157] that al-Baghdādī included in his zīj material from various earlier astronomers, viz. Yaʿqūb ibn Ṭāriq, Ḥabash al-Ḥāsib, Abu'l-Wafāʾ and Kushyār ibn Labbān, but that he also added tables by his own hand.

Below we will see that the equation of time material in the Baghdādī Zīj is based on the rules and parameters of Aḥmad ibn ʿAbdallāh Ḥabash al-Ḥāsib al-Marwazī, who worked at the Abbasid court in Baghdad around the year 830.[158] Two or three different zījes are attributed to Ḥabash al-Ḥāsib, but only two manuscripts of these zījes are extant, namely Istanbul Yeni Cami 784/2 and Berlin Ahlwardt 5750. I will refer to these manuscripts as *Ḥabash Istanbul* and *Ḥabash Berlin* respectively.[159]

3.4.2 Preliminaries

Al-Baghdādī's table for the equation of time occurs on folio 41^{v} of the manuscript Paris BN Arabe 2486. Its independent variable is "the corrected degree of the sun" (درجة الشمس المعدلة), i.e. the true solar longitude, and runs from 1° to 30° for every sign of the ecliptic. The equation of time is given in minutes and seconds of an hour and is additive from 3° Aquarius to 9° Pisces, indicated by the word يزيد ("is added") at the top of the columns, subtractive otherwise, indicated by ينقص ("is subtracted"). The table is preceded by the following explanatory text on folios 40^{v}–41^{r}:[160]

[157]See Kennedy 1957, p. 48; Kennedy 1968; and Jensen 1971/72, p. 327.

[158]More information about Ḥabash al-Ḥāsib can be found in Section 4.1.2 of this thesis or in the article by S. Tekeli in the Dictionary of Scientific Biography (*DSB*).

[159]Debarnot 1987 contains a careful study of the Istanbul manuscript and a table of contents of the Berlin manuscript.

[160]My own translation. Text between square brackets has been added.

Knowledge of the Equation of Time according to the Opinion of Ḥabash
Know that for the time of [i.e. for which we calculate] the positions of the luminaries there is a correction which is known as the equation of time. Whenever it is neglected, it leads to a perceptible difference especially in the time of the [solar and lunar] eclipses and the [visibility of the] crescent and the positions at which we need the recording of the time.

And the method of this is that you correct the sun at the time you want.[161] Next you always add to its mean position 3 degrees 34 minutes.[162] Then you examine the difference between the right ascension of the [true position of the] sun and the mean motion plus what you added to it.[163] Always multiply the difference between the two by four minutes and what comes out are minutes of an hour; keep these in mind. Then you examine: if the ascension of the corrected sun is in excess, you add the minutes of an hour that you have to the time for which you made the calculation. And if the mean sun plus its increase is in excess, we subtract the minutes from the time for which we made the calculation. And we correct the luminaries for it in particular; as far as the five planets are concerned there is no need for this.

And if you want, enter the table of the solar motion in hours with these minutes and take what is opposite it shifted one sexagesimal place to the right,[164] and add it to the mean solar motion or subtract it in accordance with what was stated and it will be [the mean solar motion] corrected for the equation of time.

Example. We assume that 4 hours of the day have passed and the totality of hours [of daylight] is 11;26 and the sun is in 18;8 Scorpio and its mean position is 229;26. We add 3;33, the result is 232;59. We take the right ascension; it is 225;38. We take the difference between them, it is 7;21, multiply it by four, the result of the multiplication is 0;29, and since the mean sun plus its increase is in excess of the right ascension, we subtract from the above-mentioned time the minutes that we have. What remains is 3;31. We correct the luminaries for it.[165]

Another Method.[166] We enter the table for solar motion in hours and their fractions with the minutes of the equation of time and take what is opposite it shifted one sexagesimal place to the right[167] and subtract it from the collected mean motion at the above-mentioned epoch [(time)].

Another Method. We enter the table with the degrees of the sun as the vertical argument and with the sign as the horizontal argument and take what we find at their intersection; they are minutes of the equation of time, and we look whether "additive" or "subtractive"

[161] What is meant is the calculation of the true solar position from the mean position by adding the solar equation. This calculation is not explained until two folios later.

[162] This is a scribal mistake for "3 degrees 33 minutes", which occurs both in the numerical example that follows and in the passages of Ḥabash's zījes mentioned below.

[163] What is meant is that the smaller of the two quantities must be subtracted from the larger.

[164] The Arabic "munḥaṭṭan" literally means "decreased". In this case it is equivalent to "divided by 60". The division is necessary since the combined subtable for the mean solar motion in hours and in fractions of an hour (i.e. in minutes) displays the mean motion for hours, whereas our present argument is given in minutes.

[165] Note that the "totality of the hours" was not necessary for the calculation of the equation of time. Furthermore, the given value for the "totality" is not found in the table for equal hours on folios 117^{v}–118^{r}, which displays $10;25,8,52^{h}$ for 18° Scorpio.

[166] This is not another method, but an example of the correction of the solar mean motion for the difference between true and mean solar time that has just been determined.

[167] See footnote 164.

is above this position and apply it.[168]
Example. We enter the table that follows and we take what is opposite the above-mentioned degree of the sun and its sign and it is 0;28,41 and there is a small difference between them.[169]

We see that al-Baghdādī instructs his reader to calculate the equation of time according to formula (3.5) in Section 3.1.1 with c taken equal to 3;33. The same rule in different words is presented in Ḥabash Istanbul, folio 91ᵛ, line 22–folio 92ʳ, line 10 and in Ḥabash Berlin, folio 17ʳ, line 2–folio 17ᵛ, line 2. Other versions of the same rule can also be found in Ḥabash Istanbul, folio 228ᵛ, lines 1–11, and in the margin of the solar equation table on folio 30ʳ of Ḥabash Berlin. The examples given by al-Baghdādī are not present in either of the manuscripts of Ḥabash's zījes, nor do we find a table for the equation of time. However, Ḥabash Berlin contains the following instructions for determining the equation of time by means of "the table".[170]

> **And as far as [the method] by means of the table is concerned**, it is [such] that we enter the column of the number [i.e. the argument] with the degree of the corrected sun and we take what is opposite it [in the column] of the sign in which the sun is, and we always multiply it by 4 minutes and we perform a calculation with addition or subtraction [according to] what we find indicated on top of the column.[171] God willing.

We conclude that the table concerned had the true solar position as its independent variable, but that the tabular values were given in equatorial degrees, contrary to al-Baghdādī's table.[172] I suspect that the equation of time material in the Baghdādī Zīj is based on the rules and parameter values of Ḥabash al-Ḥāsib, but that al-Baghdādī himself or his source compiled the explanatory text, the numerical examples and the table. Below I will perform an extensive analysis of al-Baghdādī's equation of time table and some of his numerical examples in order to investigate this hypothesis. In particular, it would be interesting to know whether the table is indeed based on parameter values attributed to Ḥabash, and whether al-Baghdādī took the equation of time material from another zīj or compiled it himself.

Before analysing al-Baghdādī's table for the equation of time, we can make the following observations concerning the underlying parameters:

- Especially in his solar and lunar tables al-Baghdādī uses Ḥabash al-Ḥāsib's parameter values (cf. Section 4.1.1). Furthermore, he copies a number of tables which we also find in Ḥabash Berlin. In his turn, Ḥabash seems to have relied on the so-called

[168] What is meant is that the equation of time value found must be added to or subtracted from mean solar time, depending on what is indicated above the value.
[169] What is meant is that there is a small difference between the result of the previous example and this one.
[170] Berlin Ahlwardt 5750, folio 17ᵛ, lines 2–5. My own translation. Text between square brackets has been added.
[171] Cf. footnote 168.
[172] As far as I know, of the extant early zījes only al-Battānī's Ṣābiʾ Zīj contains equation of time values in equatorial degrees; see Nallino 1899-1907, vol. 2, pp. 61–64. However, al-Battānī's tabular values must always be subtracted from true solar time to obtain mean solar time, whereas the instructions in Ḥabash Berlin seem to indicate that part of the table for the equation of time concerned is additive.

"Mumtaḥan observations" of the group of astronomers led by Yaḥyā ibn Abī Manṣūr, but does not seem to have participated in these observations himself.[173]

- Al-Baghdādī consistently uses the value $\varepsilon = 23;35$ for the obliquity of the ecliptic.[174] Ḥabash al-Ḥāsib uses $\varepsilon = 23;35$ in most of his tables. However, Ḥabash Istanbul contains a set of auxiliary tables based on $\varepsilon = 23;33$,[175] and a set of tables for the oblique ascension and the hour length based on Ptolemy's value $\varepsilon = 23;51,20$.[176] Furthermore, I found that the right ascension table on folios 90^v–91^v of Ḥabash Berlin involves $\varepsilon = 23;33$ and that the oblique ascension table on folios 109^r–111^v is based on $\varepsilon = 23;51,20$.[177]

- The solar equation table on folios 90^r–91^r of Ḥabash Istanbul was computed for a maximum equation of $1°59'0''$.[178] The same holds for the table on folios 30^r–31^r of Ḥabash Berlin, which gives values to sexagesimal thirds and is extremely accurate. Apparently both solar equation tables were based on a value for the solar eccentricity e close to $\text{arcSin}\,1;59,0 \approx 2;4,35,30$;[179] the independent variable of both tables is the mean solar longitude. The solar equation table on folios 45^r–47^v of the Baghdādī Zīj is identical to the one in Ḥabash Berlin and is attributed to Abu'l-Wafāʾ.

- In the Baghdādī Zīj no explicit value for the solar apogee is mentioned in the instructions for the calculation of the true solar position on folio 43^r. Instead, the solar apogee for a particular year must be calculated using the table of apogee motion on folio 42^v. This table is essentially the same as the one on folio 28^r of Ḥabash Berlin, but contains a large number of scribal and possibly other errors. I will show elsewhere that the four subtables involve three different values for the motion of the apogee. The solar apogee for the epoch 1 Hijra is given as $79°30'23''2'''43^{iv}53^{v}$.[180] As an example al-Baghdādī calculates the solar position on 4 Muḥarram 684 Hijra. Using his table

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[173] Cf. Kennedy 1956a, pp. 126–127, nos 15 and 16, and Sayılı 1960, pp. 50–87.

[174] See the results of my case study on the Baghdādī Zīj in Section 4.2.

[175] Istanbul Yeni Cami 784/2, folios 226^r–227^r; see Debarnot 1987, p. 62. The value $\varepsilon = 23;33$ of the obliquity of the ecliptic was observed by the group of astronomers that compiled the *Mumtaḥan Zīj* in the same city and period in which Ḥabash al-Ḥāsib wrote his zījes; see e.g. Caussin de Perceval 1804, pp. 54–57 (pp. 38–41 in the separatum), and the declination table in the Mumtaḥan Zīj based on this value (Escorial Ms. árabe 927, folio 53^r–54^r, or the facsimile edition Yahya ibn Abi Mansur 1986, pp. 102–104). Ibn Yūnus also confirms that Ḥabash used both 23;33 and 23;35 for the obliquity of the ecliptic in his zīj; see Schoy 1922, p. 11.

[176] Istanbul Yeni Cami 784/2, folios 134^r–147^r. The oblique ascension tables for the seven climata were taken directly from the *Handy Tables* (cf. Stahlman 1959, pp. 210–238). However, the table on folios 134^r–136^v is based on $\varepsilon = 23;51,20$ in combination with the latitude value $\phi = 34°$ of Samarra in Iraq, which does not occur in Ptolemy and for which also various examples of calculations are included in Ḥabash Istanbul (cf. Debarnot 1987, pp. 47 and 54).

[177] Unpublished results.

[178] The table was analysed in Salam & Kennedy 1967, pp. 494–495, and is highly inaccurate.

[179] Note that the solar equation cannot be computed from the maximum equation $q_{\max}$ without converting $q_{\max}$ to the eccentricity e. See Section 2.6.2 for more information about the determination of the eccentricity from the maximum equation, as well as for another example of an accurate solar equation table with maximum $1°59'0''$.

[180] Both for the sun and for the five planets the last three digits of the apogee position at epoch are written under the first three.

for the motion of the apogee, he finds the solar apogee to be in $89°32'48''$.[181]

In Ḥabash Istanbul the solar apogee is said to be in $82°39'$ both in the heading of the table for mean solar motion on folios 89^{r}–89^{v} and in the instructions for determining the true solar position on folio 165^{v}. In Ḥabash Berlin we find the same instructions including the same solar apogee value on folios 167^{v}-168^{r}. Furthermore there is a list of planetary apogee values for the epoch 1 Hijra on folio 28^{r}, which is completely identical to the one in the Baghdādī Zīj, and a hardly legible list of apogee values for the year 876 Hijra (1471 A.D.) on folio 17^{v}.

It can be checked that the planetory apogee values given in the two manuscripts of Ḥabash's zījes and in the Baghdādī Zīj are consistent with the tables for the motion of the apogee in these manuscripts. The value 82;39 of the solar apogee, which is mentioned both in Ḥabash Istanbul and in Ḥabash Berlin (see above) and which, according to Ibn Yūnus, was observed in the year 214 Hijra by the group of astronomers that compiled the Mumtaḥan Zīj,[182] differs from the epoch value 79;30,23,2,43,53 by $3°8'36''57'''16^{iv}7^{v}$. This corresponds to the motion of the apogee in approximately 214 Hijra years, as can be calculated from the table.[183]

- We have seen above that both in the Baghdādī Zīj and in the two manuscripts of Ḥabash's zījes, the epoch constant c is given as 3;33. I have not been able to find an explanation for the use of this value. It does not make the equation of time zero at the epoch 1 Hijra of al-Baghdādī's and Ḥabash's mean motion tables or at another plausible epoch based on the Hijra calendar. Furthermore, there is no reason to believe that the value derives from the use of the Seleucid Era or the Persian Yazdigird Era. In Ḥabash Istanbul various other values for the epoch constant are mentioned in a number of applications of the equation of time.[184]

3.4.3 Analysis

All values in the table for the equation of time on folio 41^{v} of the Baghdādī Zīj are displayed in Tables 3.9 to 3.12 (the "error" column will be explained below). First we note that the first differences of the table do not display any obvious pattern. Therefore we will assume that no linear or other type of interpolation was applied and we will include all tabular values in the least squares estimations carried out below.

If we assume that, in agreement with the information in the explanatory text, the true solar longitude is the independent variable of al-Baghdādī's table for the equation of

[181] This result is consistent with an example on folio 42^{v}, where al-Baghdādī calculates the apogee of Jupiter for the same date.

[182] See Caussin de Perceval 1804, p. 56 (p. 40 in the separatum).

[183] The apogee values for the five planets as given in Ḥabash Istanbul (folios 103^{r}, 107^{r}, 111^{r}, 115^{r} and 119^{r}) are also confirmed by Ibn Yūnus to derive from the "Mumtaḥan group" (see Caussin de Perceval, pp. 218–220; pp. 234–236 in the separatum). Like the solar apogee, these values differ from the respective epoch values by $3°8'36''57'''16^{iv}7^{v}$. The fact that the last four digits of all six apogee values are identical makes it highly probable that they were all computed by subtracting a constant ending in $36''57'''16^{iv}7^{v}$ from observed values accurate to minutes.

[184] See Debarnot 1987, p. 42 and the sections of the same article referred to there. I have not investigated any further the various values of the epoch constant mentioned by Ḥabash.

true solar long.	equation of time	error	true solar long.	equation of time	error	true solar long.	equation of time	error
1	0; 6,41		31	0;16,36	−1	61	0;19,56	
2	0; 7, 2		32	0;16,52		62	0;19,53	
3	0; 7,24		33	0;17, 9	+2	63	0;19,49	
4	0; 7,44	−1	34	0;17,21		64	0;19,45	
5	0; 8, 7		35	0;17,35		65	0;19,40	
6	0; 8,29	+1	36	0;17,45	−3	66	0;19,34	
7	0; 8,49	−1	37	0;18, 0	−1	67	0;19,28	
8	0; 9,12		38	0;18,13		68	0;19,21	
9	0; 9,33		39	0;18,24		69	0;19,14	
10	0; 9,55		40	0;18,35		70	0;19, 6	
11	0;10,16		41	0;18,45		71	0;18,58	
12	0;10,37		42	0;18,55		72	0;18,49	
13	0;10,59	+1	43	0;19, 3	−1	73	0;18,37	−3
14	0;11,20		44	0;19,12		74	0;18,30	
15	0;11,40		45	0;19,20		75	0;18,20	
16	0;12, 1		46	0;19,28	+1	76	0;18, 9	
17	0;12,22		47	0;19,34		77	0;17,58	−1
18	0;12,42		48	0;19,40		78	0;17,48	+1
19	0;13, 2		49	0;19,45		79	0;17,36	
20	0;13,20	−2	50	0;19,50		80	0;17,24	
21	0;13,41		51	0;19,53		81	0;17,11	
22	0;14, 1		52	0;19,57		82	0;16,59	
23	0;14,20		53	0;20, 0	+1	83	0;16,46	
24	0;14,38		54	0;20, 1		84	0;16,33	
25	0;14,56		55	0;20, 3	+1	85	0;16,21	+1
26	0;15,14		56	0;20, 3		86	0;16, 7	
27	0;15,32		57	0;20, 3		87	0;15,54	
28	0;15,49		58	0;20, 2		88	0;15,41	+1
29	0;16, 5		59	0;20, 1		89	0;15,27	
30	0;16,21		60	0;19,59		90	0;15,12	−1

Table 3.9: Table for the equation of time from the Baghdadi Zij (1st quadrant)

time, then the minimum possible standard deviation of the tabular errors is 33 sexagesimal *thirds*. If, on the other hand, we assume that the mean solar longitude is the independent variable, then the minimum possible standard deviation is 15 *seconds*. We conclude that the independent variable of the table is indeed the true solar longitude.

If we take the conversion factor D equal to 15, we find the following approximate 95 % confidence intervals for the underlying parameters by applying a least squares estimation:

parameter	*95 % confidence interval*
obliquity of the ecliptic	⟨23;34,56,30 , 23;35, 7,35⟩
solar eccentricity	⟨ 2; 4,35, 5 , 2; 4,37,36⟩
solar apogee	⟨82;38,35, 2 , 82;39,44,49⟩
epoch constant	⟨ 3;32,58, 8 , 3;32,59,51⟩

If we take D equal to 15;2,28, the approximate 95 % confidence intervals are as follows:

true solar long.	equation of time	error	true solar long.	equation of time	error	true solar long.	equation of time	error
91	0;14,59		121	0;10, 7		151	0;13,18	
92	0;14,46		122	0;10, 5		152	0;13,33	
93	0;14,32		123	0;10, 3		153	0;13,48	
94	0;14,18		124	0;10, 2		154	0;14, 3	
95	0;14, 5		125	0;10, 1		155	0;14,20	+1
96	0;13,52		126	0;10, 1		156	0;14,35	
97	0;13,39		127	0;10, 2		157	0;14,51	−1
98	0;13,26		128	0;10, 3		158	0;15, 9	
99	0;13,13		129	0;10, 5		159	0;15,27	+1
100	0;13, 0		130	0;10, 8		160	0;15,44	
101	0;12,48		131	0;10,10	−1	161	0;16, 2	
102	0;12,36		132	0;10,15		162	0;16,20	
103	0;12,24		133	0;10,19		163	0;16,37	−1
104	0;12,13		134	0;10,24		164	0;16,56	
105	0;12, 1	−1	135	0;10,30		165	0;17,15	
106	0;11,51		136	0;10,36		166	0;17,34	
107	0;11,41		137	0;10,43		167	0;17,52	−1
108	0;11,31		138	0;10,50	−1	168	0;18,12	
109	0;11,20	−1	139	0;10,59		169	0;18,31	
110	0;11,12		140	0;11, 7		170	0;18,51	
111	0;11, 4		141	0;11,17		171	0;19,10	
112	0;10,56		142	0;11,26		172	0;19,29	
113	0;10,48		143	0;11,37		173	0;19,49	
114	0;10,41		144	0;11,48		174	0;20, 8	
115	0;10,34	−1	145	0;11,59		175	0;20,28	
116	0;10,29		146	0;12,11		176	0;20,48	+1
117	0;10,23		147	0;12,23		177	0;21, 7	
118	0;10,18		148	0;12,36		178	0;21,26	
119	0;10,14		149	0;12,50		179	0;21,45	
120	0;10,10		150	0;13, 3	−1	180	0;22, 4	

Table 3.10: Table for the equation of time from the Baghdadi Zij (2nd quadrant)

parameter	*95 % confidence interval*
obliquity of the ecliptic	⟨23;36,49,23 , 23;37, 0,30⟩
solar eccentricity	⟨ 2; 4,55,34 , 2; 4,58, 6⟩
solar apogee	⟨82;38,34,57 , 82;39,44,55⟩
epoch constant	⟨ 3;33,33,10 , 3;33,34,53⟩

For $D = 15$ the historically attested parameter values $\varepsilon = 23;35$, $e = 2;4,35,30$ (derived from a maximum equation of $1°59'$), and $\lambda_A = 82;39$ are all contained in the respective 95 % confidence intervals, whereas $c = 3;33$, given in the explanatory text, falls just outside the confidence interval for the epoch constant. On the other hand, for $D = 15;2,28$ the confidence intervals for the obliquity, the eccentricity and the epoch constant are not even in the neighbourhood of attested values for these parameters. We conclude that al-Baghdādī used the conversion factor $D = 15$, in agreement with the multiplication by four minutes indicated in the explanatory text, and the attested values of the underlying

true solar long.	equation of time	error	true solar long.	equation of time	error	true solar long.	equation of time	error
181	0;22,23		211	0;29, 6	+2	241	0;25,48	
182	0;22,42		212	0;29, 8		242	0;25,29	
183	0;23, 0		213	0;29,12		243	0;25, 9	
184	0;23,19		214	0;29,16		244	0;24,49	
185	0;23,37		215	0;29,18		245	0;24,28	
186	0;23,54	−1	216	0;29,20		246	0;24, 7	
187	0;24,12		217	0;29,20	−1	247	0;23,45	
188	0;24,30		218	0;29,22		248	0;23,22	
189	0;24,47		219	0;29,21		249	0;22,59	
190	0;25, 3		220	0;29,20		250	0;22,35	
191	0;25,20		221	0;29,18		251	0;22,10	
192	0;25,36	+1	222	0;29,14	−1	252	0;21,45	
193	0;25,51		223	0;29,11		253	0;21,19	
194	0;26, 6		224	0;29, 7		254	0;20,53	
195	0;26,21		225	0;29, 2		255	0;20,27	
196	0;26,35		226	0;28,56		256	0;20, 0	
197	0;26,49		227	0;28,49		257	0;19,32	
198	0;27, 2		228	0;28,41		258	0;19, 4	
199	0;27,15		229	0;28,32		259	0;18,36	
200	0;27,27		230	0;28,23		260	0;18, 8	
201	0;27,39		231	0;28,13		261	0;17,39	
202	0;27,50		232	0;28, 2		262	0;17,10	
203	0;28, 1		233	0;27,50		263	0;16,40	
204	0;28,11		234	0;27,38		264	0;16,11	
205	0;28,21		235	0;27,24		265	0;15,41	
206	0;28,29		236	0;27,10		266	0;15,11	
207	0;28,37		237	0;26,55		267	0;14,41	
208	0;28,45		238	0;26,39		268	0;14,13	+2
209	0;28,51	−1	239	0;26,23		269	0;13,41	
210	0;28,58		240	0;26, 6		270	0;13,11	

Table 3.11: Table for the equation of time from the Baghdadi Zij (3^{rd} quadrant)

parameters given above.

The "error" columns in Tables 3.9 to 3.12 display the differences between al-Baghdādī's table for the equation of time and a recomputation using the parameter values found above. We can see that the agreement is very good. There are only 48 differences, of which 26 are $-1''$ and 14 are $+1''$. No evident clustering or symmetry can be recognized in the errors and the conditions necessary for applying the least squares estimation (namely that the tabular errors are independent and that they have zero mean and identical standard deviations) can be assumed to hold.

It seems plausible that al-Baghdādī or his source did not compute the equation of time values to their full accuracy. The right ascension values could readily be taken from a table, but since the apogee value 82;39 is not a round number, some type of interpolation was probably used to calculate the solar equation values $q(\lambda)$ from a table for the solar equation $q(a)$ as a function of the true solar anomaly ($a = \lambda - \lambda_A$). Another possibility is that $q(\lambda)$ was calculated by means of inverse interpolation in a table for the solar equation

true solar long.	equation of time	error	true solar long.	equation of time	error	true solar long.	equation of time	error
271	0;12,41		301	0; 0,17		331	−0; 1,27	
272	0;12,11		302	0; 0, 1		332	−0; 1,18	
273	0;11,41		303	−0; 0,13		333	−0; 1, 9	
274	0;11,11		304	−0; 0,27		334	−0; 0,59	
275	0;10,41		305	−0; 0,40		335	−0; 0,47	+1
276	0;10,11	−1	306	−0; 0,53	−1	336	−0; 0,37	
277	0; 9,43		307	−0; 1, 3		337	−0; 0,29	−4
278	0; 9,14		308	−0; 1,14		338	−0; 0,12	
279	0; 8,45		309	−0; 1,24		339	−0; 0, 1	−2
280	0; 8,16		310	−0; 1,32		340	0; 0,14	−1
281	0; 7,48		311	−0; 1,40		341	0; 0,29	
282	0; 7,20		312	−0; 1,47		342	0; 0,44	
283	0; 6,53		313	−0; 1,54		343	0; 0,59	
284	0; 6,26		314	−0; 1,59		344	0; 1,15	
285	0; 6, 0		315	−0; 2, 4		345	0; 1,32	
286	0; 5,34		316	−0; 2, 7		346	0; 1,48	
287	0; 5, 8		317	−0; 2,10		347	0; 2, 6	
288	0; 4,43		318	−0; 2,12		348	0; 2,23	
289	0; 4,19		319	−0; 2,14		349	0; 2,41	
290	0; 3,54	−1	320	−0; 2,14		350	0; 3, 0	
291	0; 3,32		321	−0; 2,14		351	0; 3,18	
292	0; 3, 9		322	−0; 2,13		352	0; 3,37	
293	0; 2,47		323	−0; 2,11		353	0; 3,57	
294	0; 2,26		324	−0; 2, 8		354	0; 4,17	
295	0; 2, 6	+1	325	−0; 2, 4		355	0; 4,37	
296	0; 1,45		326	−0; 2, 0		356	0; 4,56	−1
297	0; 1,26		327	−0; 1,55		357	0; 5,17	
298	0; 1, 8		328	−0; 1,49		358	0; 5,38	
299	0; 0,50		329	−0; 1,42		359	0; 5,59	
300	0; 0,33		330	−0; 1,35		360	0; 6,19	−1

Table 3.12: Table for the equation of time from the Baghdadi Zij (4$^{\text{th}}$ quadrant)

$\bar{q}(\bar{a})$ as a function of the mean solar anomaly. Unfortunately no further information about the method of calculation can be obtained, since neither the errors in the equation of time values themselves nor those in the extracted right ascension and solar equation show any patterns.[185] Only the following can be said:

- By comparing the errors in al-Baghdādī's table for the equation of time with those in the extracted right ascension and solar equation values, it can be noted that all but one of the errors larger than $1''$ in absolute value are probably *not* the result of an error in the underlying tables, but must have been introduced during the calculation of the equation of time values. The pairs of errors with absolute value $1''$ for arguments 6

[185] See Section 3.1.3 on how to extract the underlying right ascension and solar equation from a table for the equation of time as a function of the true solar longitude. The maximum error in the extracted values will be $7\frac{1}{2}''$ provided that the equation of time values used for the extraction are correct. In this case 18 out of 180 extracted right ascension values and 19 out of 180 extracted solar equation values contain an error larger than $7\frac{1}{2}''$.

and 186, 115 and 295, and 176 and 356 could be the result of erroneous solar equation values; the pairs for arguments 37 and 217, 88 and 268, and 155 and 335 could be due to incorrect right ascension values.[186]

- The right ascension table on folios 141ᵛ–143ʳ and the solar equation table on folios 45ʳ–47ᵛ of the Baghdādī Zīj are too accurate to produce the number of errors that we find in the table for the equation of time.[187] In fact, any recomputation involving right ascension and solar equation values accurate to seconds leads to practically correct equation of time values to seconds and therefore approximately to the same number of differences between al-Baghdādī's equation of time and the recomputation. This also holds if linear interpolation or inverse linear interpolation in a solar equation table is involved.

3.4.4 Conclusions

The table for the equation of time on folio 41ᵛ of the Baghdādī Zīj was computed in full agreement with the information found in the explanatory text just preceding it, i.e. according to the rules and parameter values of Ḥabash al-Ḥāsib. The independent variable is the true solar longitude and the conversion factor 15. The underlying parameter values are 23;35 for the obliquity of the ecliptic, 2;4,35,30 for the solar eccentricity, 82;39 for the solar apogee, and 3;33 for the epoch constant.

For three reasons it seems plausible that al-Baghdādī took the equation of time material in his zīj from another source. Firstly, there seems to be no relation between the table for the equation of time and other tables in the Baghdādī Zīj. In particular, the equation of time is less accurate than we would expect from the accuracy of the right ascension and solar equation tables. Secondly, Ḥabash's apogee value $\lambda_A = 82;39$ was completely out of date by the time al-Baghdādī wrote his zīj. As was indicated above, al-Baghdādī calculates the solar apogee for the year 684 Hijra to be in $89°32'48''$ in an example two folios after the table for the equation of time. Thirdly, the numerical example for the determination of the equation of time assumes a true solar longitude of $228°8'$, whereas all other examples related to the calculation of solar, lunar and planetary positions use the dates 15 Rabīʿ I 683 Hijra (22 June 1284 A.D.) and 4 Muḥarram 684 Hijra (12 March 1285), which fall just before the date of completion mentioned in the colophon of the zīj on folio 255ʳ, and on which solar positions are given as $\bar{\lambda} = 78°7'22''$ and $\lambda = 0°1'51''$ respectively.

An extensive investigation of the tables in Ḥabash Istanbul and, in particular, Ḥabash Berlin will be necessary to find out whether al-Baghdādī took his equation of time material from one of the zījes of Ḥabash al-Ḥāsib himself. On the one hand, we have seen that the instructions in Ḥabash Berlin for using a table for the equation of time presuppose a

[186] Note that, if the underlying right ascension satisfies the symmetry relations mentioned in Section 3.1.3, an error $e(\lambda)$ in the right ascension value for argument λ ($\lambda \in \langle 0, 90\rangle$) leads to errors $\frac{1}{15}e(\lambda)$ in the equation of time values for arguments λ and $180+\lambda$, to errors $-\frac{1}{15}e(\lambda)$ in the values for arguments $180-\lambda$ and $360-\lambda$. On the other hand, an error $f(\lambda)$ in the solar equation value for argument λ ($\lambda \in \langle 0, 180\rangle$) leads to an error $\frac{1}{15}f(\lambda)$ in the equation of time value for argument λ and to an error $-\frac{1}{15}f(\lambda)$ in the value for argument $180+\lambda$.

[187] Cf. Section 4.3.8 of this thesis.

table with values in equatorial degrees,[188] whereas al-Baghdādī's equation of time values are given in hours. On the other hand, the Baghdādī Zīj contains various tables which are also found in Ḥabash Berlin, at least part of which can be assumed to derive from Ḥabash himself.[189]

3.5 Results

An extensive technical explanation of the equation of time as it was tabulated by Ptolemy and by many Islamic astronomers was presented in Section 3.1.1. The little that is known about the computation of ancient and mediaeval tables for the equation of time was summarized (Section 3.1.2) and various powerful methods for analysing such tables were discussed (Section 3.1.3). Finally, extensive analyses of the tables for the equation of time in Ptolemy's *Handy Tables* and in the Greek papyrus London 1278 (Section 3.2), in Kushyār ibn Labbān's *Jāmiʿ Zīj* (Section 3.3) and in the zīj by al-Baghdādī (Section 3.4) were carried out. In all four cases we managed to find the mathematical formula used for the calculation of the table and the values of the four underlying parameters. The results were as follows:

author (period)	Ptolemy (c. 150)	Kushyār (c. 1000)	al-Baghdādī (c. 1285)
independent variable	λ	$\bar{\lambda} + 2$	λ
obliquity of the ecliptic	23;51,20	23;35	23;35
solar eccentricity	2;30	2;4,45	2;4,35,30
solar apogee	66;0	84;0	82;39
epoch constant	3;34,7,30	4;4	3;33
conversion factor	15	15	15

The table in the papyrus London 1278 is not included in this table, since we found that it was probably computed by subtracting the values in the *Handy Tables* from the constant 0;32 and rounding to minutes. Thus the underlying parameter values are the same except for the epoch constant.

We have seen that Ptolemy simplified his calculations by using linear interpolation between right ascension values for every 10 degrees of solar longitude. Furthermore, although all historical sources concerning the *Almagest* and the *Handy Tables* give the value $65°30'$ for the solar apogee, Ptolemy rounded this value to $66°0'$ in his table for the equation of time, probably in order to avoid interpolation between his solar equation values.

Kushyār ibn Labbān simplified the use of his planetary tables by applying so-called "displaced equations", which must always be added to (or always subtracted from) the mean motion. His table for the equation of time was also adapted to this principle and actually had the mean solar position plus 2 as its argument.

[188] See Section 3.4.2 above.
[189] Cf. Sections 4.1.1 and 4.2 of this thesis.

Al-Baghdādī followed Ḥabash al-Ḥāsib's rules for the determination of the equation of time precisely, both in his explanatory text and in the computation of his table. The underlying value for the solar apogee is Ḥabash's 82°39′, almost seven degrees smaller than the current value in the time of al-Baghdādī. This and the fact that the table for the equation of time is less accurate than we would expect from the right ascension and solar equation tables in the Baghdādī Zīj make it probable that the table and the explanatory text were taken from an earlier source, possibly from Ḥabash himself.

The tables for the equation of time that we investigated use three different types of epochs. The epoch 1 Philip of the *Handy Tables* was such that the equation of time practically always had to be added to mean solar time to obtain true solar time, and the epoch constant was chosen so as to make the minimum equation zero. On the other hand, the epoch 1 Nabonassar of the equation of time values in the papyrus London 1278 was such that the equation of time practically always had to be subtracted from true solar time. Kushyār made his equation of time always subtractive as well, but at his epoch 1 Yazdigird the equation was far from zero. We do not know how Kushyār compensated for the error that he made in this way. The epoch constant of al-Baghdādī's table for the equation of time does not seem to be related to the epoch of his planetary tables. In most cases his equation of time values must be subtracted from mean solar time, in some cases they must be added.

The number of tables analysed in this chapter is too small to allow us to draw general conclusions about the calculation of tables for the equation of time by Greek and Islamic astronomers and about the historical development of the methods of calculation. The following questions have been touched upon but remain essentially open:

1. Did Islamic astronomers tend to compute their own tables for the equation of time based on current values of the underlying parameters (especially the solar apogee changed rapidly through the centuries) or did they also copy tables from earlier sources?

2. On the basis of which criteria did mediaeval astronomers choose the independent variable of their table for the equation of time? Was this choice related to the way in which the solar tables of the zīj under consideration had to be used, or was it based mainly on considerations of ease of computation?

3. Which were the various methods that were used to fix the epoch constant? Is there a relation between the method chosen and the way in which the solar tables in the zīj concerned had to be used? If the equation of time did not approximately reach its minimum or maximum value at epoch, how was the difference accounted for?

4. Which other astronomers beside al-Kāshī made use of the accurate value 15;2,28 for the conversion factor? A preliminary analysis showed that Ulugh Beg's table for the equation of time, in the computation of which al-Kāshī may very well have been involved, uses the conversion factor 15;2,28 as well. On the other hand, the table for the equation of time by al-Ṭūṣī, who influenced al-Kāshī in many ways, was based on $D = 15$.

Chapter 4

Case Study: the Baghdādī Zīj

4.1 Introduction

4.1.1 Preliminaries

The manuscript Paris Bibliothèque Nationale Arabe 2486 is the only extant copy of a zīj in Arabic written by Jamāl al-Dīn Abī al-Qāsim ibn Maḥfūẓ al-munajjim al-Baghdādī. The manuscript was copied in the month Muḥarram of the year 684 Hijra (March/April 1285) and is written in a very clear hand. The punctuation of the Arabic is generally correct and the manuscript contains very few marginal notes and additions by later owners or users. Since, as far as I know, the zīj of al-Baghdādī was not referred to by later astronomers either, it seems probable that it was not widely used. Nevertheless, for historians the zīj is very interesting, because, as we will see below, it contains material deriving from various important early Islamic astronomers. Since the name of al-Baghdādī's zīj cannot be determined with certainty (folio 1^r shows a nicely ornamented, but apocryphal title *al-zīj al-waqibiya* in Kufic script[1]), I will follow Kennedy and King and will simply refer to the zīj as the *Baghdādī Zīj*. Plate 4.1 displays two pages from the Baghdādī Zīj.[2] The table on these pages is attributed to al-Baghdādī himself and will be analysed in Section 4.3.14.

Until 1957 confusion existed with respect to the period in which the Baghdādī Zīj was compiled. In *Kashf al-Ẓunūn*, Hajji Khalīfa states that al-Baghdādī lived during the reign of the caliph al-Muqtadir, who died in 932.[3] Suter was the first to cast doubt upon this claim, since he found no information about al-Baghdādī in the *Fihrist* by Ibn al-Nadīm or in the *Ta'rīkh al-ḥukamā'* by Ibn al-Qifti.[4] After his *Zīj Survey* had been published in 1956, Kennedy investigated the Baghdādī Zīj and concluded from the text in the colophon that the manuscript had been copied for al-Baghdādī personally.[5] Since the zīj contains many numerical examples involving dates during the years 1284 and 1285,[6]

[1]Cf. de Slane 1883–1895, pp. 440–441.
[2]The size of a page in the original is 26 × 18 cm.
[3]See Flügel 1835–1858, vol. 3, p. 559.
[4]See Suter 1900, p. 197, no. 490.
[5]See Kennedy 1957, p. 48. The colophon occurs on folio 255^v.
[6]See Kennedy 1968, p. 130 and the examples mentioned in Section 3.4 of this thesis.

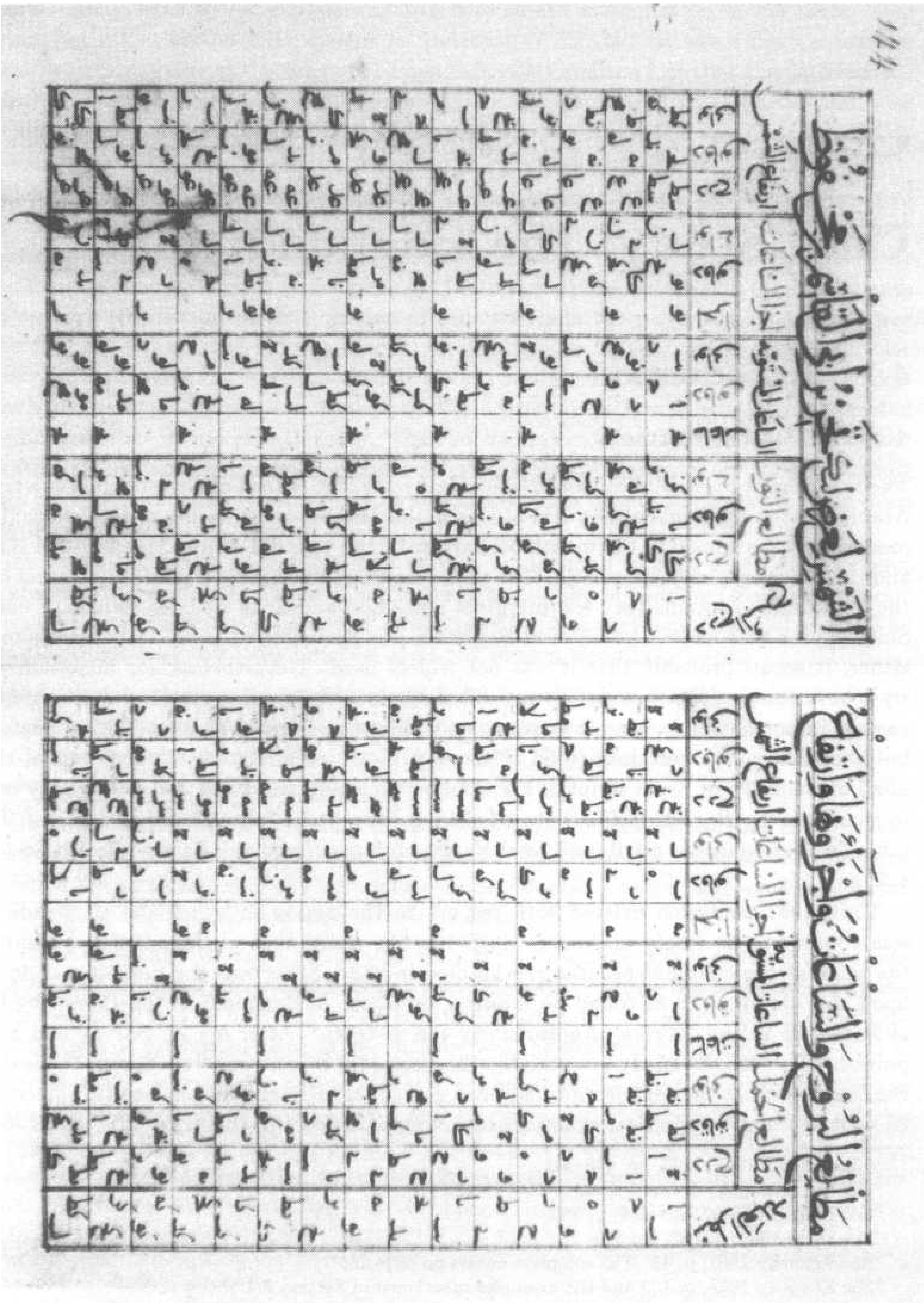

Figure 4.1: Fragment of the Baghdādī Zīj (Paris BN Arabe 2486, folios 143^v and 144^r)

we conclude that al-Baghdādī finished compiling his zīj in the year 1285. One other work by al-Baghdādī is known, namely the "Treatise on the Operation of the Astrolabe", a fragment of which is extant in the manuscript London British Museum 1002/24.[7]

The colophon of the Baghdādī Zīj states that the author used important elements from the zījes of earlier astronomers and added numerical examples of his own. This has been confirmed by the investigations concerning the zīj that have been conducted so far. Kennedy already noted that "one of the tables" in the Baghdādī Zīj (namely the solar equation table) is attributed to Abu'l-Wafāʾ.[8] He also studied the material on the visibility of the lunar crescent, which is attributed to Yaʿqūb ibn Ṭāriq.[9]

In the most extensive publication on the Baghdādī Zīj so far, Jensen investigates the three procedures presented by al-Baghdādī for determining the longitude of the moon.[10] He finds that the first procedure is attributed to Ḥabash al-Ḥāsib and that the accompanying tables are identical to those in the Istanbul version of Ḥabash's zīj.[11] Contrary to the first procedure, the second procedure takes the inclination of the lunar orbit with respect to the ecliptic into account. This requires an additional table, which turns out to be identical to the corresponding table in the Mumtaḥan Zīj by Yaḥyā ibn Abī Manṣūr. Jensen remarks that also al-Baghdādī's solar eclipse theory is practically identical to Yaḥyā's theory. The third procedure is probably an original achievement by al-Baghdādī which drastically reduces the number of tables and operations needed in order to determine the lunar longitude.

Finally, the material in the Baghdādī Zīj on the positions of comets, on the Uighur calendar and on the computation of Easter, as well as the table for the tangent of declination (see Section 4.3.7) have been briefly described.[12]

My own preliminary investigations revealed that, like the lunar tables, the solar tables in the Baghdādī Zīj are essentially based on the rules and parameter values of Ḥabash al-Ḥāsib. In fact, the solar mean motion table is computed for Ḥabash's parameter value 0;59,8,20,35,25°/day. The table for the motion of the apogee, which turns out to be based on three different values for the daily motion, is essentially identical to the table in the Berlin version of Ḥabash's zīj. In Section 3.4 of this thesis it is shown that al-Baghdādī's table for the equation of time was also computed on the basis of Ḥabash's parameter values, including the value 82°39′ for the solar apogee, which was completely out of date by the time al-Baghdādī compiled his zīj.

In this chapter most of the trigonometrical and spherical astronomical tables in the zīj of al-Baghdādī will be analysed. Together the analyses of these tables constitute an extensive overview of methods, both straightforward and more advanced, that can be

[7]See Suter 1900, p. 197, no. 490.

[8]See Kennedy 1957, p. 48. Biographical and bibliographical information concerning Abu'l-Wafāʾ can be found in Section 4.1.2 below.

[9]See Kennedy 1968. New light was shed on Yaʿqūb ibn Ṭāriq's table for lunar crescent visibility in Hogendijk 1988b.

[10]See Jensen 1971–1972.

[11]Biographical and bibliographical information concerning Ḥabash al-Ḥāsib can be found in Section 4.1.2 below.

[12]See Kennedy 1957, pp. 48–49; Kennedy 1964, p. 443; Saliba 1970, p. 193; and Lesley 1957, p. 127.

used in order to determine the mathematical structure of mediaeval astronomical tables. It will be shown how these methods can yield detailed information about the method of computation of certain tables. Furthermore, it will be demonstrated that the information thus found can be successfully used to investigate the origin of the tables concerned.

For every type of table that will be analysed Section 4.3 first presents a short definition of the tabulated function introducing the parameters, some mathematical properties (such as the presence of symmetry) and methods for determining the mathematical structure and underlying parameter values. Next, the lay-out of each table is briefly described and the tabular values are extensively analysed by means of the above-mentioned methods. The order in which the tables are discussed is approximately the order in which I assume that the tables were calculated. Thus I start with the sine table, which is the basis of all trigonometrical calculations, and end with the oblique ascension tables, which were computed by subtracting the equation of daylight from the right ascension. I will refer to the tables from the Baghdādī Zīj as "al-Baghdādī's tables" even though many of these tables were probably borrowed from earlier sources. The following Section 4.1.2 presents biographical and bibliographical information concerning the astronomers from whom al-Baghdādī copied some of his tables. Section 4.1.3 explains in more detail how the gathered information concerning al-Baghdādī's tables has been organized. In Section 4.1.4 all techniques are described which are used for the analysis of the tables in Section 4.3. In a later publication I hope to include a complete table of contents of the Baghdādī Zīj, relevant information found in the explanatory text and analyses of the planetary tables in the zīj.

4.1.2 Information about Ḥabash, Abu'l-Wafāʾ and Kushyār

As we will see in Sections 4.2 and 4.3, al-Baghdādī probably copied most of his trigonometric and spherical astronomical tables from the early Islamic astronomers Ḥabash al-Ḥāsib, Abu'l-Wafāʾ al-Būzjānī and Kushyār ibn Labbān. This section gives information on the life and works of these astronomers.

Ḥabash al-Ḥāsib

Aḥmad ibn ʿAbdallāh Ḥabash al-Ḥāsib al-Marwazī was a native of Marw in Turkestan.[13] From 825 till 835 he worked in Baghdad as an astronomer in the service of the Abbasid caliphs al-Maʾmūn and al-Muʿtaṣim. Although Ḥabash al-Ḥāsib is known to have made observations, there is no evidence that he cooperated with the group of astronomers headed by Yaḥyā ibn Abī Manṣūr that compiled the Mumtaḥan Zīj in the same period.

Two manuscripts of zījes written by Ḥabash are extant, namely Istanbul Yeni Cami 784/2 (to be referred to as *Ḥabash Istanbul*) and Berlin Ahlwardt 5750 (to be referred

[13] Most of the following information is taken from the *DSB*-article "Ḥabash al-Ḥāsib" by S. Tekeli and from Debarnot's article (1987) about the two extant manuscripts of Ḥabash's zījes.

to as *Ḥabash Berlin*).[14] Both manuscripts were copied in the 13th century. Most of the material in the Istanbul manuscript is likely to have been part of Ḥabash's original zīj. According to Kennedy, the introduction was clearly written by Ḥabash himself.[15] The Berlin manuscript, however, is probably a later recension. Some of the tables are identical to tables in Ḥabash Istanbul, others can clearly be recognized as later additions. Occasionally tables are attributed to other early Islamic astronomers such as Thābit ibn Qurra (fl. 870) and al-Nairīzī (fl. 900); in other cases the tables are so much more accurate than the corresponding tables in the Istanbul manuscript that it seems unlikely that they were computed by Ḥabash.

My own preliminary analyses of a number of spherical astronomical tables in Ḥabash Istanbul and Ḥabash Berlin showed that these manuscripts involve a large variety of parameter values. For example, in both manuscripts we find for the obliquity of the ecliptic the values 23;33 (associated with Yaḥyā ibn Abī Manṣūr), 23;35 (the most common Islamic value) and 23;51,20 (the Ptolemaic value).

It is unclear how many zījes Ḥabash al-Ḥāsib wrote. Later Islamic biographers and astronomers refer to Ḥabash's zījes by many different names. Debarnot assumes that the Istanbul manuscript contains what was later referred to as the "Zīj of Ḥabash". According to Debarnot, this zīj was also named Damascene or Arabic Zīj.[16] The latter title served to distinguish the zīj from another zīj by Ḥabash based on Indian methods. Kennedy mentions the possibility that Ḥabash wrote a third zīj, namely the Ṣaghīr (Little) Zīj.[17]

Many publications deal with aspects of the zījes of Ḥabash al-Ḥāsib. Here I will only mention the article by Kennedy describing the parallax material in both the Istanbul and Berlin manuscripts[18] and the article by Salam and Kennedy on the tables for the solar and lunar equations.[19] An extensive study by Irani of Ḥabash's famous *jadwal al-taqwīm* (auxiliary table) has not yet been published.[20] Other publications of interest are indicated by an asterisk in the bibliography of Debarnot 1987.

We have already seen that al-Baghdādī frequently incorporated in his zīj rules and parameter values deriving from Ḥabash. In particular, al-Baghdādī's solar and lunar tables are based on Ḥabash's parameter values, but most of these tables were probably computed anew, either by al-Baghdādī or by his source. In the remainder of this chapter we will see that the Baghdādī Zīj contains twelve tables with values to sexagesimal thirds, three of which can also be found in Ḥabash Berlin. I will argue that these three tables probably were *not* part of Ḥabash's original zīj.

[14] The Istanbul manuscript is described and analysed in Kennedy 1956a, p. 127 (no. 16) and pp. 153–154, and is extensively investigated in Debarnot 1987. The Berlin manuscript is summarized in Ahlwardt 1893, pp. 200–203 and is described and analysed in Kennedy 1956a, pp. 126–127 (no. 15) and pp. 151-153. Debarnot 1987, pp. 63–65 gives a list of passages in the Berlin manuscript that also occur in Istanbul.

[15] See Kennedy 1956a, p. 127 (no. 16). The introduction was translated in Sayılı 1955.

[16] See Debarnot 1987, p. 37. In my opinion, Debarnot's motivation is not very convincing since for each title she relies on a single tabular value or parameter value mentioned by a later astronomer.

[17] See Kennedy 1956a, pp. 126 (no. 15) and 131 (no. 39).

[18] See Kennedy 1956b, pp. 42–43.

[19] Salam & Kennedy 1967.

[20] Irani 1956.

Abu'l-Wafā᾽ al-Būzjānī

Muḥammad ibn Muḥammad ibn Yaḥyā ibn Ismāʿil ibn al-ʿAbbās Abu'l-Wafā᾽ al-Būzjānī was born in the Persian region Būzjān in 940 and died in Baghdad in 997 or 998.[21] After he moved to Baghdad in 959, Abu'l-Wafā᾽ wrote important works on arithmetic, trigonometry and astronomy. His "Book on What is Necessary from the Science of Arithmetic for Scribes and Businessmen" was used on a large scale. Abu'l-Wafā᾽ gave new solutions to many problems in spherical trigonometry and computed trigonometric tables with an accuracy that had not been achieved until his time.[22] Finally, he made astronomical observations and wrote two astronomical handbooks, the Wāḍiḥ Zīj and al-Majisṭī (Almagest). A substantial part of al-Majisṭī is extant in a Paris manuscript.[23] This unique copy contains the trigonometric material referred to above, but does not contain any tables. The Wāḍiḥ Zīj is not extant in any form.[24]

The relation between the tables in the Wāḍiḥ Zīj and those belonging to al-Majisṭī is unknown. More information about Abu'l-Wafā᾽'s tables must be obtained from zījes that have incorporated material from his works, such as the Baghdādī Zīj. Part of a sine table attributed to Abu'l-Wafā᾽ can be found in a recension of the Mumtaḥan Zīj, the original of which was written more than a century before Abu'l-Wafā᾽ by a group of astronomers at the court in Baghdad headed by Yaḥyā ibn Abī Manṣūr.[25] A solar equation table attributed to Abu'l-Wafā᾽ occurs in the Baghdādī Zīj.[26] Kennedy found that various later zījes such as the anonymous Shāmil Zīj and the Athīrī Zīj by al-Abharī incorporated Abu'l-Wafā᾽'s mean motion parameters.[27]

In Sections 4.3.1 and 4.3.3 below we will see that various sine and cotangent values given by Abu'l-Wafā᾽ in the extant part of al-Majisṭī are equal to the values found in al-Baghdādī's sine and cotangent tables. Furthermore, al-Baghdādī's table for the equation of daylight will be shown to have been computed by means of inverse linear interpolation in a sine table with accurate values to four sexagesimal places for every $15'$ of the argument, and Abu'l-Wafā᾽ is known to have computed an accurate sine table with just that format. It is thus possible that, in addition to the solar equation, the Baghdādī Zīj contains more of Abu'l-Wafā᾽'s tables.

[21] Much of the information about Abu'l-Wafā᾽ presented here was taken from the *DSB*-article by A.P. Yuschkevich. See also Suter 1900, no. 167, pp. 71–72.

[22] Abu'l-Wafā᾽'s spherical trigonometry is described in Delambre 1819, pp. 156–163 and Carra de Vaux 1892. Information about Abu'l-Wafā᾽'s trigonometric tables can be found in Delambre 1819, pp. 157-158 and in Schoy 1923, pp. 393–394.

[23] Paris Bibliothèque Nationale Ms. 2494 (107 folios, 12th century). See de Slane 1883–95, p. 442.

[24] See Kennedy 1956a, p. 134 (no. 73) and Suter 1900, pp. 71–72.

[25] See the manuscript Escorial Ms. árabe 927, folios 51v–52r or the facsimile edition Yaḥyā ibn Abī Manṣūr, pages 99–100.

[26] This solar equation table occurs on folios 45^{r}–47^{v} of the Baghdādī Zīj. The same table without attribution can be found in the zīj of Ḥabash, Berlin Ahlwardt 5750, folios 30^{r}–31^{r}. The table was analysed in Kennedy & Salam 1967, pp. 493–494. Some remarks about the accuracy and the underlying eccentricity value of the table are made in Section 2.6.2 of this thesis.

[27] See Kennedy 1956a, pp. 134 (no. 73), 129 (no. 29) and 133 (no. 56). See furthermore Section 2.6.2 of this thesis for more information about the Shāmil Zīj and the Athīrī Zīj.

Kushyār ibn Labbān

Kushyār ibn Labbān ibn Bāshahrī Abu'l-Ḥasan al-Jīlī flourished around the year 1000 in Baghdad.[28] The attribute al-Jīlī indicates that Kushyār was a native of the region Jīlān in northern Iran. Kushyār's main achievements were in the fields of arithmetic, trigonometry and astronomy. He wrote a work "The Elements of Hindu Reckoning" about sexagesimal arithmetic and computed extensive trigonometric tables.[29] In his astronomical works Kushyār made use of the parameters of al-Battānī (c. 900) instead of making his own observations. As was shown in Section 3.3 of this thesis, Kushyār made his planetary tables easier to use by applying so-called "displaced equations".

It is unclear whether Kushyār wrote one or two astronomical handbooks. In "The Book of the Astrolabe" he mentions the *Jāmiʿ Zīj* ("Comprehensive Astronomical Tables") and the *Bāligh Zīj* ("Extensive Astronomical Tables") as two different works. Kennedy suggests that the Bāligh Zīj is an abridged version of the Jāmiʿ.[30] I made a cursory analysis of the tables in four manuscripts of Kushyār's zīj(es): Istanbul Fatih 3418, Berlin Ahlwardt 5751, Leiden Or. 8 (1054), and Cairo Dār al-Kutub Mīqāt 188/2.[31] The oldest of these manuscripts, Fatih 3418, is entitled "The Book of the Jāmiʿ Zīj" and is divided into four treatises containing instructions, tables, explanations and proofs respectively. The same division is found in the Berlin and Leiden manuscripts, although the third and fourth treatises are not actually present in the Berlin manuscript. From the given tables of contents and from the coherence of the material in the Istanbul, Berlin and Leiden manuscripts, it can be concluded that, except for the appended tables described below, both explanatory text and tables in the three manuscripts were part of the original zīj written by Kushyār.[32] The Cairo manuscript contains only a number of Kushyār's tables.

My analysis revealed that all four manuscripts contain essentially the same set of somewhat more than 50 tables that are listed in the tables of contents referred to in footnote 32. There are, however, small differences between the manuscripts, which may be due to the existence of two different zījes by the hand of Kushyār ibn Labbān. In Section 3.3.3 of this thesis the difference is described between the right ascension tables in the Istanbul and Cairo manuscripts on the one hand and the Berlin and Leiden manuscripts on

[28] Saidan comes to this conclusion because Kushyār was not mentioned in Ibn al-Nadīm's *Fihrist* (c. 995), but was first referred to by al-Bayhaqī (died in 1065) in *Tatimma ṣiwān al-ḥikma*; see the article "Kushyār" in the Dictionary of Scientific Biography (*DSB*).

[29] These tables for sine, versed sine, tangent and cotangent can be found in all four manuscripts of Kushyār's zīj(es) mentioned below.

[30] Kennedy 1956a, p. 125 (nos 7 and 9).

[31] The Istanbul manuscript was copied in the year 1150 and seems to contain the Jāmiʿ Zīj in its original form. The Berlin manuscript is described in Ahlwardt 1893, pp. 203–206, which also gives an extensive table of contents. The Leiden manuscript is analysed by Kennedy in his *Zīj Survey* (Kennedy 1956a, pp. 156–157). The Cairo manuscript is described in King 1986b, p. 45 (no. B70). *GAS*, vol. 5, pp. 247–248 mentions six more manuscripts that contain fragments of the Jāmiʿ Zīj, but these manuscripts do not contain Kushyār's tables.

[32] The table of contents of the explanatory text can be found in Fatih 3418, folios 1^v–2^v; Berlin Ahlwardt 5751, pp. 2–4 (also given in Ahlwardt 1893, pp. 204–205) and Leiden Or. 8 (1054), folios 1^v–2^v. The list of tables can be found in Fatih 3418, folio 37^v; Berlin Ahlwardt 5751, p. 35 (also given in Ahlwardt 5751, p. 205) and Leiden Or. 8 (1054), folio 21^r.

the other. We also find two different sine tables in Kushyār's zīj(es), one with arguments $1, 2, 3, \ldots, 90°$, which is found in the Istanbul, Cairo and Berlin manuscripts,[33] and one with arguments $d°m'$ for every $d = 0, 1, 2, \ldots, 89$ and $m = 0, 1, 2, \ldots, 15, 18, 21, \ldots, 60$, which is found in the Berlin and Leiden manuscripts.[34] Further research is necessary in order to establish whether Kushyār in fact wrote two different zījes.

As far as the date of compilation of the Jāmiʿ Zīj is concerned, it is mentioned in Section 3.3 that Kushyār gives his planetary apogee values for the year 962, whence it seems plausible that he compiled his zīj(es) shortly after this date. This is confirmed by a reference in *GAS*, which indicates that from one of the manuscripts of the Jāmiʿ Zīj it can be concluded that Kushyār finished his zīj in 964.[35]

At the end of the Berlin and Leiden manuscripts of the Jāmiʿ Zīj we find a large number of tables which apparently were not part of Kushyār's original work. In many cases these tables display functions which can also be found in the main set of tables. In the Berlin manuscript, a number of planetary equation tables are attributed to Ibn al-Aʿlam (c. 960), some other tables to Abū Maʿshar (Albumasar, c. 850). In the Leiden manuscript, a set of planetary equation tables is taken from the Fākhir Zīj by al-Nasawī (c. 1030), some other tables mention al-Bīrūnī (c. 1000) as their author. However, most of the appended tables in both manuscripts are not attributed. In Section 2.6.3 of this thesis it is shown that one of the appended tables in the Berlin manuscript, which displays the true solar position as a function of the mean solar position, is based on an approximate method of computation and parameter values which are elsewhere attributed to Yaḥyā ibn Abī Manṣūr.

In this Chapter we will see that various tables from the main set of tables of Kushyār's Jāmiʿ Zīj were copied by al-Baghdādī. However, Kushyār's work was not one of al-Baghdādī's major sources. Only if no tables from other sources were available for a certain function or certain parameter values did al-Baghdādī make use of the Jāmiʿ Zīj.

4.1.3 Organization of this Chapter

This chapter contains various types of information concerning al-Baghdādī and the trigonometric and spherical astronomical tables in his zīj. The chapter is organized in such a way that the information of every type can be read independently. Thus the reader who is interested in biographical and bibliographical information should turn to the "Preliminaries" (Section 4.1.1). The reader who has a general interest in the methods of computation used by al-Baghdādī and in the relation between the Baghdādī Zīj and other zījes should first read the overview of all results in Section 4.2. If his curiosity has then been aroused, he will be able to find more detailed information about the tabulated function, the lay-out and the analysis of a particular table in Section 4.3. Finally, the reader who is interested in the amount and the type of scribal errors found in al-Baghdādī's zīj is referred to the

[33]See Istanbul Fatih 3418, folio 41^{v}; Cairo DM 188/2, folio 13^{r} and Berlin Ahlwardt 5751, page 42.
[34]See Berlin Ahlwardt 5751, pages 49–63 and Leiden Or. 8 (1054), folios 24^{v}–31^{r}.
[35]See *GAS*, vol. 5, pp. 343–344.

Apparatus (Section 4.4). In a later publication I hope to include a complete table of contents of the Baghdādī Zīj.

In more detail, the contents of the four sections into which this chapter is divided are as follows:

1. Sections 4.1.1 and 4.1.2 contain biographical and bibliographical information concerning al-Baghdādī and three astronomers who influenced his zīj.
2. Section 4.2 collects all results of the analyses of the trigonometric and spherical astronomical tables in the Baghdādī Zīj. These results are classified according to: accuracy, underlying parameters, computational techniques, and dependence on other zījes.
3. Section 4.3 presents technical information for every type of table under the heading **definition**. This information includes: the modern formula for the tabulated function, the underlying parameters, the method according to which mediaeval astronomers computed the table, mathematical properties such as symmetry, and methods for the analysis of the table.
4. Section 4.3 presents information about the lay-out of every table under the heading **description**. This information includes folio numbers, column numbers, titles and relevant textual information that can be found in the manuscript. Furthermore, the results of the analysis of each table are briefly summarized. Information concerning the format of each table can be found in the so-called "error statistics table" that occurs immediately after the description and also contains statistical data regarding the errors in the table. The information concerning the format of the table includes the range of the arguments and the unit. (Remember that the unit of a table is the greatest common divisor of the tabular values; see Section 1.1.3). The abbreviations 2E and 4E in the "arguments" column of the error statistics table indicate a table with double entries and quadruple entries respectively. (For instance, in a table with double entries for the solar equation as a function of the mean solar anomaly, every tabular value serves two arguments, λ and $360 - \lambda$; cf. Section 1.1.3.)
5. Section 4.3 presents an extensive analysis of every table under the heading **analysis.** The analyses demonstrate a large variety of techniques that can be used to correct scribal errors and to recover the mathematical structure of the types of tables concerned. The amount of advanced mathematics and statistics involved is relatively small. An explanation of most of the techniques used can be found below in Section 4.1.4. In each case $T(x)$ will denote the tabular value for argument x of the table under consideration.

   Some of the numerical results of the analyses are presented in the above-mentioned "error statistics table". These results include:

   - The values of the underlying parameters (here R denotes the radius of the base circle for the trigonometric tables, ε the obliquity of the ecliptic and ϕ the geographical latitude).
   - The total number of tabular values used for the estimation of the parameter values and for the final recomputation (this number is indicated by N). Note that very often we cannot use all tabular values for the estimation of the underlying parameter

values because of possible dependencies between the tabular errors, e.g. as a result of the symmetry of the table. Whenever N equals 90, I used the tabular values from the first quadrant for the estimation of the parameters. If N equals 180, I used the values from the first and second quadrants.

- The total number of differences between the table under consideration and a modern recomputation on the basis of the parameter values found (this number is indicated by n). In all cases the scribal errors indicated in the Apparatus were corrected before the recomputation was performed. In each instance the rounding of the recomputed values was performed in the modern way.
- The mean of the differences between table and recomputation (indicated by μ).
- The standard deviation of the differences between table and recomputation (indicated by σ).

6. The Apparatus in Section 4.4 presents outliers, differences between al-Baghdādī's tables and recomputed tables, and corrections of scribal errors. In all cases it is specified how the presented information was obtained. Note that only in incidental cases are the corrections of scribal errors based on a comparison with copies of al-Baghdādī's tables in other manuscripts. In most cases the corrections were made by verifying the symmetry of the tables, by inspecting the tabular differences, or by comparing the table with a preliminary recomputation (see the explanation of these techniques in Section 4.1.4).

I have decided not to include in this thesis the tabular values of all tables from the Baghdādī Zīj that I analysed or the errors in those tables. However, the tables are available on diskette (MS-DOS compatible) both in ASCII and in the format of my computer program TABLE-ANALYSIS (see Section 1.4.1). By means of this program all tables can easily be recomputed for any desired values of the underlying parameters or they can be printed in various lay-outs. Furthermore, the parameter estimations that I performed and other estimations can be carried out and the conditions regarding the tabular errors can be tested. Both the program TABLE-ANALYSIS and the diskette with all tables investigated in this chapter can be purchased from the author.

4.1.4 Techniques of Analysis

For the analysis of the trigonometric and spherical astronomical tables in the Baghdādī Zīj the following techniques (here presented in the order in which they will usually be applied) are available:

- Correction of scribal and computational errors that can be recognized as large deviations from the symmetry of the table. In most tables the symmetry has clearly been utilized for the computation. Consequently, practically all deviations from the symmetry are probably due to scribal mistakes. **Example.** The right ascension $\alpha(\lambda)$ satisfies the symmetry $\alpha(180-\lambda) = 180-\alpha(\lambda)$ for every λ. If a given table for the right ascension displays the values 20;19,50 for argument 22 and 159;40,53 for argument 158, then it is very probable that the correct value for argument 22 was 20;19,7 and that the erroneous digit 50 is the result of a scribal error (ر →ن ; note that the correction 53 → 10 (نج→ى) is less plausible).

- Correction of scribal errors that can be recognized as large deviations from the general pattern in the differences between the table and a (preliminary) recomputation. Since practically all tabulated functions are smooth, these differences are expected to be smooth as well, even if the parameter values or the function used for the recomputation are not correct.
  Example. If $-26'', -25'', -26'', +14'', -27'', -26'', -26''$ are consecutive differences between a given table and a preliminary recomputation, the difference $+14''$ very probably corresponds to a scribal error of $-40''$ in the tabular value concerned.

- Correction of scribal errors by inspection of tabular differences. A scribal error E in a tabular value superimposes $k+1$ consecutive errors of respective sizes $(-1)^i \binom{k}{i} E$ $(i = 0, \ldots, k)$ on the k-th order finite differences of the table. Thus the first order differences will contain two consecutive errors $+E$ and $-E$ and the fourth order differences will contain five consecutive errors $+E, -4E, +6E, -4E, +E$. Scribal errors can be corrected by locating such error patterns in the finite order differences. Since the rounding errors of the table propagate in the same fashion, only relatively large scribal errors can be recovered in this way. It can be seen that the k-th order finite differences of a correct table of a smooth function rapidly approach 0 if k increases. It turns out that in practice the error patterns can be recognized particularly easily in the third or fourth order differences.
  Example. Consecutive first order differences $59'9'', 59'20'', 59'30'', 1°0'21'', 59'11''$, $1°0'3'', 1°0'14'', 1°0'25''$ and the corresponding fourth order differences $+2'', -1'', -1''$, $+42'', -2'42'', +4'3'', -2'43'', +41'', 0'', -1'', +3''$ point to a scribal error of $+40''$.

- Investigation of the possible use of linear or higher order interpolation by inspection of tabular differences. k-th order interpolation can be recognized by groups of (practically) constant k-th order finite differences separated by jumps in these differences. An example of the use of this technique can be found in Section 4.3.13.2.

- Reconstruction of tables that were used for the computation of the table to be investigated. In certain cases, for instance the oblique ascension, the underlying tables can be "extracted" by utilizing the symmetry of the table; this technique is explained under the **definition** heading of Section 4.3.13. In other cases the underlying tables can be computed directly from the table to be investigated once we have determined (some of) the underlying parameter values and the presumable method of computation. We can analyse the reconstructed tables separately in order to recover their mathematical structure and underlying parameter values.

- Estimation of the underlying parameters of the table to be investigated. An extensive discussion of parameter estimation can be found in Chapter 2; brief, informal explanations of the four estimators that I use are presented in Section 2.1. In the case of the Baghdādī Zīj the estimation of the underlying parameter values was straightforward for practically all tables.

- Interpretation of the patterns in the differences between the table to be investigated and a recomputation for the correct parameter values. From such patterns conclusions can be drawn about the use of (inverse) interpolation, the accuracy of the underlying tables and the type of errors in these tables. On the basis of the conclusions drawn, we can try to recompute the table under consideration precisely, for instance by using other tables from the Baghdādī Zīj or from earlier zījes and, if necessary, by performing rounding or truncation at intermediate steps of the calculation, or by applying interpolation or inverse interpolation between available tabular values.

The techniques described here can be applied to many different types of tables. Most of the techniques are incorporated in my computer program Table-Analysis described in Section 1.4.1.

4.2 Summary of Results

In Section 4.3 I will analyse most of the trigonometric and spherical astronomical tables in the Baghdādī Zīj.[36] I will classify the results according to various aspects of the computation and origin of the tables: accuracy, underlying parameters, computational techniques, and dependence on earlier zījes. It turns out to be convenient to distinguish two groups of tables:

Group 1.

- Sine (folios 224^{v}–225^{v})
- Versed Sine (226^{r}–227^{r})
- Tangent / Cotangent (227^{v}–228^{v})
- Solar Declination (129^{v}–130^{v}, 1st col.)
- Solar Altitude (119^{v}–120^{r})
- Tangent of Declination (235^{r})
- Right Ascension (141^{v}–143^{r})
- Sine of the Equation of Daylight (235^{v})
- Equation of Daylight (117^{r})
- Length of Daylight (117^{v}–118^{r})
- Hour Length (118^{v}–119^{r})
- Oblique Ascension (139^{v}–141^{r})

[36] So far I have not yet extensively investigated the following spherical astronomical tables: two tables concerning prayer times (folios 120^{v}–122^{v}); a table for the ascensions of the zodiacal signs and for the length of the longest day as a function of the geographical latitude (folios 149^{v}–150^{r}); a table for the so-called *aṣl* (folio 236^{r}). The tables for the ascensions and for the length of the longest day appear to be based on the Ptolemaic value of the obliquity. The table for the aṣl (defined by aṣl $\lambda = \cos\phi \cdot \cos\delta(\lambda)$, where $\delta(\lambda)$ is the solar declination) has accurate values to sexagesimal thirds and is based on obliquity 23;35 and latitude 33;25. Hence this table probably belongs to the first group defined below.

Group 2.

- Cotangent ($R = 7$ / $R = 12$, folio 229^r)
- Second Declination (129^v–130^v, 3rd col.)
- Oblique Ascension for latitudes from 30° to 40° (150^v–160^r)
- Oblique Ascension, Length of Daylight, Hour Length and Solar Altitude for geographical latitude 32°20′, attributed to al-Baghdādī (143^v–149^r)

We will see that the tables of the first group are strongly related with respect to their method of computation and origin. The tables all have the same number of sexagesimal fractional digits and most of the tables can be shown to have been computed from other tables in the group. It seems probable that all tables of the first group were computed by the same astronomer and I will conjecture that this astronomer was Abu'l-Wafāʾ al-Būzjānī.

The tables of the second group are not as strongly related as the tables of the first group. We will see that some of the tables of the second group were computed by al-Baghdādī himself, others he borrowed from Kushyār ibn Labbān. In general the tables of the second group are less accurate than those of the first group. The latitude value 32°20′ found in the table attributed to al-Baghdādī is typical for him.

Accuracy

All tables of the **first group** have three sexagesimal fractional digits. Only in the case of the *sine* and the *versed sine* is the final sexagesimal digit generally accurate. The *tangent* values on folios 227^v–228^v contain rapidly increasing errors for arguments approaching 90°; the *cotangent* values on the same folios are identical to the corresponding tangent values. The final sexagesimal digits of the other nine tables of the first group contain small errors with standard deviations varying from $3'''$ to $23'''$. Below we will see that the tables of the first group are directly related. Consequently, the sizes of the errors in the tables are correlated as well. In general, the errors become larger as the number of steps necessary to compute the tables from the ultimately underlying sine table increase.

As far as the **second group** is concerned, the *cotangent* tables on folio 229^r and the *oblique ascension* tables on folios 150^v–160^r have a single sexagesimal fractional digit. The values of the cotangent tables and of the oblique ascension tables for geographical latitudes $30°, 31°, 33°, 34°, 35°$ and 37° are generally correct. The oblique ascension tables for latitudes $36°, 38°, 39°$ and 40° contain many errors of 1 or 2 minutes. The remaining tables of the second group display two sexagesimal fractional digits. In the case of the *oblique ascension* table on folios 143^v–149^r, which is attributed to al-Baghdādī, the given seconds are highly inaccurate because of the use of interpolated values to minutes for the equation of daylight. These values were shown to underlie the *length of daylight* and *hour length* tables on the same folios as well.

Underlying parameter values

The underlying parameter values of the trigonometric and spherical astronomical tables in the Baghdādī Zīj present very few surprises. Except for the cotangent tables on folio 229^{r}, which display the length of the shadow cast by gnomons of 12 digits and 7 feet respectively, the radius of the base circle of the trigonometric functions is taken equal to 60 units. The value for the obliquity of the ecliptic used in all spherical astronomical tables that I analysed is $\varepsilon = 23;35$. In each instance the underlying value of the geographical latitude is in agreement with the value mentioned in the tabular heading. The tables of the **first group**, which all have three sexagesimal fractional digits, are based on the common value $\phi = 33;25$ for the latitude of Baghdad. This value was observed by Abu'l-Wafāʾ in the year 987,[37] and was included in the geographical tables of Ibn Yūnus and al-Bīrūnī in the same period.[38] The table on folios 143^{v}–149^{r}, which is attributed to al-Baghdādī, was computed for the value $\phi = 32°20'$ mentioned in the heading. Of the cities to which this latitude value is assigned in geographical tables, the one to which al-Baghdādī can most plausibly be connected is Wāsiṭ, 220 kilometres south-east of Baghdad.[39]. The zījes by Kushyār ibn Labbān and al-Bīrūnī both give the latitude value $32°20'$ for Wāsiṭ.[40] In al-Baghdādī's own geographical table on folios 162^{v}–163^{v} the latitude value $32°20'$ does not occur for any locality and the value assigned to Wāsiṭ is $32°30'$.[41]

Methods of computation

As far as the method of computation of the investigated tables from the Baghdādī Zīj is concerned, it can be noted that the tables of the **first group**, which all have values to sexagesimal thirds and are intended for the latitude of Baghdad, are strongly dependent. In fact, most of the tables in this group can be shown to be computed directly from others according to methods which are consistent both in type and accuracy. Although some of the links between the tables could not be recovered, we conclude that, possibly apart from the sine and the tangent, all tables in the first group were computed by the same astronomer. The following details of the computation of the tables of the first group were revealed (all tables referred to are those of the first group):

1. The *versed sine* was computed directly from the sine table.
2. The *tangent* and *cotangent* tables on folios 227^{v}–228^{v} were computed from the sine table by calculating quotients $60 \cdot \mathrm{Sin}\, x/\mathrm{Sin}\,(90 - x)$.
3. The *solar altitude* was computed by adding the values from the declination table to 56;35.

[37] See Abu'l-Wafāʾ's *al-Majisṭī* in the manuscript Paris Bibliothèque Nationale 2494, folio 20^{r} or Delambre 1819, p. 156.

[38] See Kennedy & Kennedy 1987, pp. 436–441 for Ibn Yūnus' table and pp. 450–460 for al-Bīrūnī's table.

[39] See Kennedy & Kennedy 1987, p. 679.

[40] Kennedy & Kennedy 1987, p. 374.

[41] See Kennedy & Kennedy 1987, pp. 521–524. Kennedy found that al-Baghdādī's geographical table was derived from al-Battānī; see Kennedy & Kennedy 1987, p. xvii.

4. The *sine of the equation of daylight* was computed from the tangent of declination by multiplying by a value for $\tan 33°25'$ accurate to at least four sexagesimal places.
5. The *equation of daylight* was computed from the sine of the equation of daylight by means of inverse linear interpolation in a sine table with values for every $15'$ of the argument which were accurately calculated to precisely four sexagesimal places.
6. The *length of daylight* and the *hour length* were both computed directly from the equation of daylight by adding 90 and dividing by $7\frac{1}{2}$ and 6 respectively.
7. The *oblique ascension* was computed by subtracting the equation of daylight from the right ascension.

In general, whenever the type of rounding used could be recovered, it was found to be modern rounding.

As far as the tables in the **second group** are concerned, we found the following details of the methods of computation:

1. The *cotangent* table for $R = 12$ was computed from a sine table having three accurate sexagesimal places.
2. The *cotangent* table for $R = 7$ was computed from the table for $R = 12$ by multiplying by 0;35 and rounding in the modern way.
3. Of the set of *oblique ascension* tables on folios 150^{v}–160^{r}, the tables for latitudes $30°, 31°, 33°, 34°, 35°$ and $37°$ were accurately computed. The tables for latitudes 38°, 39° and 40° were computed from accurate right ascension values and from values for the equation of daylight determined by means of linear interpolation between accurate values for arguments $6k + 1$ (integer k). The table for latitude 36° was computed from values for the right ascension and for the equation of daylight which were both determined by means of linear interpolation; the nodes, occurring for multiples of 6° of the argument, are accurate in the case of the equation of daylight but contain a number of errors in the case of the right ascension.
4. The *oblique ascension*, *length of daylight* and *hour length* tables for geographical latitude $32°20'$, which can be found on folios 143^{v}–149^{v}, were all computed from the same inaccurate values for the equation of daylight displayed in Table 4.2 on page 192. These values were probably determined by means of linear interpolation. The right ascension used for the computation of the oblique ascension table contained many errors of $\pm 1''$. The *solar altitude* table on the same folios was probably calculated from the declination table of the first group.

Dependence on earlier zījes

None of the tables in the Baghdādī Zīj analysed in Section 4.3 is explicitly attributed to earlier authors. Because of the coherence of the tables in the **first group**, it seems probable that these tables were all computed by the same astronomer. We found that the *equation of daylight* and *hour length* tables of the first group can also be found in the zīj of Ḥabash al-Ḥāsib extant in the manuscript Berlin Ahlwardt 5751. For this reason and because of the fact that the table of the second group attributed to al-Baghdādī is much less sophisticated than the tables of the first group, we conclude that al-Baghdādī was *not* the author of the tables in the first group. Since we have seen in Section 4.1.2 that Ḥabash Berlin in its turn is highly inhomogeneous, we cannot automatically conclude that Ḥabash was the author of the equation of daylight and hour length tables of the first group. In Section 4.1.1 we have seen that various tables in the Baghdādī Zīj (in particular those relating to solar and lunar theory) are based on the rules and parameter values of Ḥabash. However, the tables of the first group are so much more accurate than most of the tables in Ḥabash's zījes that it seems unlikely that they were computed by Ḥabash.

I conjecture that the tables in the Baghdādī Zīj belonging to the first group derive from one of the zījes written by Abu'l-Wafāʾ al-Būzjānī (cf. Section 4.1.2). It can be noted that the solar equation table in Ḥabash Berlin, which, like the tables of the first group, has three sexagesimal fractional digits and contains very few errors, can also be found in the Baghdādī Zīj and is there attributed to Abu'l-Wafāʾ. As will be indicated in Sections 4.3.1, 4.3.3 and 4.3.10, there is reason to believe that al-Baghdādī took his sine and cotangent values from Abu'l-Wafāʾ. (Note, however, that the sine table in the Mumtaḥan Zīj attributed to Abu'l-Wafāʾ is not the same as the one in the Baghdādī Zīj; see Section 4.3.1. This could be explained by assuming that al-Baghdādī copied his sine table from Abu'l-Wafāʾ's al-Majisṭī, whereas the Mumtaḥan Zīj contains the table originally found in the Wāḍiḥ Zīj.) Finally, Abu'l-Wafāʾ observed the latitude value for Baghdad which is consistently used for the tables in the first group.

In the near future I hope to make an extensive analysis of the manuscript Ḥabash Berlin in order to find more information about the origin of its tables and hence about the origin of the tables in the Baghdādī Zīj belonging to the first group.

As far as the tables of the **second group** are concerned, we found that the cotangent tables on folio 229^r, the second declination table, and the oblique ascension table for latitude 36° were taken from Kushyār ibn Labbān's Jāmiʿ Zīj. The table on folios 143^v–149^r is explicitly attributed to al-Baghdādī and we have found no reason to question this attribution. In fact, the underlying latitude value $32°20'$ can be considered to be typical for al-Baghdādī. As we have seen, the accuracy of the oblique ascension, length of daylight and hour length in this table is not impressive. The same holds for the set of oblique ascension tables on folios 150^v–160^r, in particular for the tables for latitudes $38°, 39°$ and 40°. These tables were shown to be based on values for the equation of daylight computed by means of a strange type of linear interpolation (see above). I suppose that al-Baghdādī compiled the set of oblique ascension tables himself, taking the table for latitude 36° from Kushyār's Jāmiʿ Zīj and possibly some of the remaining tables from other, hitherto unidentified sources.

Summarizing the above information, we conclude that al-Baghdādī copied the accurate trigonometric and spherical astronomical tables in his zīj from earlier works. I conjecture that his source were the non-extant tables from Abu'l-Wafāʾ's al-Majisṭī. Incidentally al-Baghdādī took less accurate tables from Kushyār ibn Labbān's Jāmiʿ Zīj, and he computed some other tables himself. To a certain extent al-Baghdādī made use of the accurate tables that he copied from earlier works for his own computations.

4.3 Description and Analysis of the Tables

The following symbols will be used in the so-called "error statistics tables", which contain information about the lay-out of the tables in the Baghdādī Zīj and the errors they contain:

R	radius of the base circle for trigonometric functions
ε	obliquity of the ecliptic
ϕ	geographical latitude
N	total number of tabular values used for the recomputation
n	total number of errors in these tabular values
μ	mean of the errors
σ	standard deviation of the errors

Special symbols are used in the error statistics tables for the oblique ascension. These symbols will be explained in Section 4.3.13.

4.3.1 Sine (folios 224^{v}–225^{v})

Definition. Mediaeval astronomers usually tabulated the trigonometric functions sine, versed sine, tangent and cotangent for a value of the radius of the base circle different from the modern value 1. Thus they tabulated a function $\operatorname{Sin} x$ defined by $\operatorname{Sin} x = R \cdot \sin x$, where in most cases the radius R was taken equal to 60, in some cases equal to 150 or 1. The value of R can directly be recovered from any sine table since we have $\operatorname{Sin} 90° = R$. In the remainder of this chapter I will assume that the radius of the base circle is equal to 60 unless otherwise stated.

Description. The Baghdādī Zīj contains a sine table on folios 224^{v}–225^{v}, which is headed "Table of the sine". We will see that this table, which was computed for radius of the base circle 60 and has values to sexagesimal thirds, is extremely accurate. Al-Baghdādī's sine table seems to be independent of other sine tables with four sexagesimal places found in extant early zījes.[42] In the remainder of this Chapter we will see that there is some evidence that al-Baghdādī's sine values derive from Abu'l-Wafāʾ.

folios	arguments	unit	R	N	n	μ	σ
224^{v}–225^{v}	$1, 2, \ldots, 90$ (4E)	0;0,0,1	60	90	9	$+1^{\text{iv}}$	19^{iv}

Analysis. From the tabular value 60;0,0,0 for argument 90 it follows that al-Baghdādī's sine table was computed for radius of the base circle 60. By comparing the tabular values with recomputed values we can correct six obvious scribal errors (see Section 4.4.1 of the Apparatus). There remain only nine small computational errors, namely for arguments $19, 22, 44, 67$ and 80 ($+1'''$) and for arguments $25, 35, 56$ and 78 ($-1'''$).

[42] As was explained in Section 1.1.3, I will speak of the "total number of sexagesimal places" of the values of a trigonometric table instead of "the number of sexagesimal fractional digits", so that I need not specify the radius of the base circle on each occasion. Thus sine values with three sexagesimal fractional digits will be said to have four sexagesimal places if $K = 60$, three if $K = 1$.

I have investigated whether the sine table in the Baghdādī Zīj could be related to the sine tables found in early Islamic zījes. Only al-Bīrūnī's *Masudic Canon* and Ibn Yūnus' *Ḥākimī Zīj* contain sine tables with values to the same number of sexagesimal places. Al-Bīrūnī gives sine values for every $15'$ of the argument calculated for radius of the base circle 1. His values for integer degrees contain nine errors of ± 1 unit, five of which are equal to errors in al-Baghdādī's table.[43] In the Ḥākimī Zīj Ibn Yūnus gives sine values for every $10'$ of the argument. His values for integer degrees and al-Baghdādī's values seem to differ in many cases; generally Ibn Yūnus' values are less accurate.[44] Abu'l-Wafāʾ is known to have computed a sine table for $R = 1$ with values to sexagesimal fourths for every $15'$ of the argument.[45] A couple of correct sine values given in Abu'l-Wafāʾ's al-Majisṭī are equal to the corresponding values in al-Baghdādī's table. The relation between al-Baghdādī's sine table and the other sine tables mentioned above needs to be investigated further. Results presented in Sections 4.3.3 and 4.3.10 will provide some evidence that al-Baghdādī took his tables for the sine and the cotangent from Abu'l-Wafāʾ.

4.3.2 Versed Sine (folios 226^{r}–227^{r})

Definition. The versed sine, denoted by Vers, is given by the modern formula

$$\operatorname{Vers} x = R \cdot (1 - \cos x), \tag{4.1}$$

where R is the radius of the base circle. In most cases the versed sine was tabulated for arguments in both the first and the second quadrants. The values in these quadrants satisfy the symmetry relation $\operatorname{Vers} x = 2R - \operatorname{Vers}(180 - x)$. The value of R can easily be determined from a table for the versed sine, since we have $\operatorname{Vers} 90^\circ = R$ and $\operatorname{Vers} 180^\circ = 2R$. A table for the versed sine can be computed from a sine table calculated for the same radius of the base circle by means of the relations $\operatorname{Vers} x = R - \operatorname{Sin}(90 - x)$ for $x \in [0, 90]$ and $\operatorname{Vers} x = R + \operatorname{Sin}(x - 90)$ for $x \in [90, 180]$.

Description. The Baghdādī Zīj contains a table for the versed sine on folios 226^{r}–227^{r}, which is entitled "Table of the versed sine". This table turns out to be computed from the sine table on the preceding folios, which was analysed in Section 4.3.1.

folios	arguments	unit	R	N	n	μ	σ
226^{r}–227^{r}	$1, 2, \ldots, 180$	0;0,0,1	60	90	9	-1^{iv}	19^{iv}

Analysis. From $T(90) = 60;0,0,0$ and $T(180) = 120;0,0,0$ it follows that al-Baghdādī's table for the versed sine is based on the value 60 for the radius of the base circle. By using the symmetry mentioned above and by comparing the manuscript values with recomputed values, we can correct eight scribal errors (see Section 4.4.2 of the Apparatus). It turns out that the corrected tabular values are completely identical to values for the versed sine recomputed from al-Baghdādī's sine table, which occurs on the preceding folios 224^{v}–225^{v}.

[43] Here I used al-Bīrūnī's sine table as published in Schoy 1927, pp. 675–677.

[44] Here I made use of the selected entries from Ibn Yūnus' sine table in the Ḥākimī Zīj presented in King 1972, p. 84 (Table 10a). In his *Kitāb al-jayb* Ibn Yūnus filled in the intermediate tabular values by means of interpolation in order to obtain sine values for every minute of the argument.

[45] See Schoy 1923, pp. 393–394.

4.3.3 Tangent and Cotangent

Definition. Mediaeval Islamic astronomers computed tables for the tangent $\text{Tan}\,x \stackrel{\text{def}}{=} R \cdot \tan x$ and the cotangent $\text{Cot}\,x \stackrel{\text{def}}{=} R \cdot \cot x$ for various values of the radius of the base circle R. Many zījes contain a tangent and/or cotangent table for $R = 1$ or $R = 60$, and furthermore cotangent tables for $R = 12$ and $R = 7$, displaying as a function of the solar altitude the length of the shadow cast by gnomons of length 12 digits (*aṣābiʿ*) and 7 feet (*aqdām*) respectively. The radius of the base circle underlying a tangent or cotangent table can easily be determined, since we have $\text{Tan}\,45° = R$ and $\text{Cot}\,45° = R$ for every value of R. Because of the relation $\text{Tan}\,x = \text{Cot}\,(90-x)$, which holds for every x, tangent and cotangent tables contain essentially the same tabular values. Hence they are here treated in a single section.

A tangent table can be computed from a given sine table by calculating the quotients $R\,\text{Sin}\,x/\text{Sin}\,(90-x)$ for every argument x. A table computed in this way can often easily be recognized, since the errors in the tangent values tend to increase rapidly as the argument approaches 90°.[46] Below we will see that the errors in all tangent and cotangent tables in the Baghdādī Zīj in fact have this property.

4.3.3.1 Tangent and Cotangent for $R = 60$ (folios 227ᵛ–228ᵛ)

Description. The Baghdādī Zīj contains a cotangent and tangent table for radius of the base circle 60 in two adjacent columns on folios 227ᵛ–228ᵛ. The columns are headed "the second shadow [computed] by somebody, [R =]60" and "the first, reversed shadow [computed] by somebody, [R =]60" respectively. From the indication "by somebody" we may conclude that al-Baghdādī copied the table from another source, but did not know who the author was. The cotangent values for arguments smaller than 45° are given to sexagesimal thirds, those for arguments larger than 45° to fourths. The reverse holds for the tangent values. It turns out that the cotangent and tangent columns contain the same values and that these values were computed from al-Baghdādī's sine table on folios 224ᵛ–225ᵛ. Two incorrect cotangent values given in Abu'l-Wafāʾ's al-Majisṭī are identical to the corresponding values in al-Baghdādī's cotangent table.

type	folios	arguments	unit	R	N	n	μ	σ
cotangent	227ᵛ–228ᵛ, 1°	$1, 2, \ldots, 44$	0;0,0,1	60	44	44	$+22'''$	$2''$
		$45, 46, \ldots, 89$	0;0,0,0,1		45	44	$+1^{\text{iv}}$	28^{iv}

Analysis. By comparing the cotangent and tangent columns of the table on folios 227ᵛ–228ᵛ we find that the two contain essentially the same values. I made use of the tangent column to correct the scribal errors in the cotangent and vice versa (see the Apparatus, Sections 4.4.3.1 and 4.4.3.2). Below I will analyse only the cotangent table. The error statistics for the tangent table are equal to those for the cotangent.

First we note that the cotangent table is based on radius of the base circle 60, since $T(45) = 60;0,0,0,0$. A recomputation shows that neither in the first half of the table

[46] The same holds for cotangent values as the argument approaches 0°.

(where the values are given to thirds) nor in the second half (where they are given to fourths) is the last sexagesimal digit accurate. Furthermore the absolute value of the errors increases rapidly from approximately $6'''$ to $10''$ as the argument decreases from $13°$ to $1°$. As was explained above, this points to calculation from sine values all having the same number of sexagesimal places. The obvious candidate for the underlying sine table is the table that occurs on folios 224^{v}–225^{v} of the Baghdādī Zīj itself (see Section 4.3.1). Indeed a recomputation of al-Baghdādī's cotangent table from his sine table by means of the formula $\mathrm{Cot}\, x = R\, \mathrm{Sin}\,(90 - x)/\mathrm{Sin}\, x$, shows an extremely good agreement: there are practically no absolute differences larger than 1 unit. The only outlier $T(49) =$ 52;9,25,55,16, which differs from the recomputed value by $-1'''1^{\mathrm{iv}}$, can be explained from a scribal mistake. Disregarding this outlier the error statistics for the first half of the table are $N = 44$, $n = 13$, $\mu = -5^{\mathrm{iv}}$, $\sigma = 36^{\mathrm{iv}}$; for the second half of the table we obtain $N = 45$, $n = 16$, $\mu = -16^{\mathrm{v}}$, $\sigma = 52^{\mathrm{v}}$. It can be checked that any sine table displaying four sexagesimal digits which does not contain the nine errors that we found in al-Baghdādī's table leads to a clearly poorer recomputation of the cotangent.

It would be useful to compare al-Baghdādī's cotangent table with the cotangent tables of al-Bīrūnī and Ibn Yūnus, as we did for the sine. The material on the cotangent in Abu'l-Wafāʾ's al-Majisṭī provides some evidence that al-Baghdādī copied his cotangent table from this work.[47] In fact, the cotangent table in al-Majisṭī is known to have had values for every $15'$ of the argument which have four sexagesimal places for arguments smaller than $45°$, five places for arguments larger than $45°$, just like al-Baghdādī's table. Furthermore, two incorrect cotangent values presented by Abu'l-Wafāʾ in examples are identical to the corresponding values in the Baghdādī Zīj.

4.3.3.2 Cotangent for $R = 12$ and $R = 7$ (folio 229^{r})

Description. Immediately following the two tables described above, we find on folio 229^{r} a table called "the second shadow where the measure [i.e. the radius of the base circle] is twelve digits and seven feet". The argument column is headed "arc of the [solar] elevation" and runs from 1 to 90. The table displays the cotangent for $R = 12$ and for $R = 7$ and has values calculated to minutes. We will see that the same table can be found in Kushyār ibn Labbān's Jāmiʿ Zīj. The column for $R = 12$ was computed from an accurate sine table with three sexagesimal places, such as can be found in the Jāmiʿ Zīj. The column for $R = 7$ was not computed directly from a sine table, but from the column for $R = 12$ by multiplying by $\frac{7}{12}$ and rounding in the modern way.

type	folios	arguments	unit	R	N	n	μ	σ
cotangent	229^{r}, $1°$	$1, 2, \ldots, 90$	0;1	12	90	6	0	$30''$
cotangent	229^{r}, $2°$	$1, 2, \ldots, 90$	0;1	7	90	19	$+3''$	$30''$

[47] See Delambre 1819, pp. 157–158.

Analysis. The cotangent tables for radii of the base circle 12 and 7, which occur on folio 229r of the Baghdādī Zīj, can also be found in the Jāmiʿ Zīj of Kushyār ibn Labbān. By comparing al-Baghdādī's tables with the copies in the available manuscripts of the Jāmiʿ Zīj,[48] we can correct five scribal errors in the table for radius 12, four in the table for radius 7. All corrections are given in the Apparatus, Sections 4.4.3.3 and 4.4.3.4.

A recomputation of the table for radius 12 shows only seven errors, but these include a couple of large errors for arguments close to 0°: the errors for arguments 1 to 5 are $-3', +1', 0, +1'$ and $-1'$ respectively. It turns out that a recomputation from sine values correctly computed to three sexagesimal places reproduces these errors precisely. We find only two differences between al-Baghdādī's table and the recomputation, namely $T(7) =$ 97;47 versus the recomputed value 97;44 and $T(13) =$ 51;58 versus 51;59. Note that the error in $T(7)$, which also occurs in all four copies in the Jāmiʿ Zīj, can be explained as a scribal error (مر → مد).

A recomputation of al-Baghdādī's cotangent table for radius 7 shows an error pattern similar to what we found for the table for radius 12. However, there are 17 differences of $\pm 1'$ between the cotangent table and a recomputation based on correct sine values with three sexagesimal places, which are distributed over the whole range of the argument. It can be shown that the author of the cotangent table for radius 7 did not use the full accuracy of the quotients $R\,\mathrm{Sin}\,(90-x)/\mathrm{Sin}\,x$, but simply multiplied the cotangent values for radius 12 by $\frac{7}{12}$ and then rounded in the modern way.[49] We find only one difference between al-Baghdādī's cotangent table for radius 7 and a recomputation following this procedure, namely $T(81) =$ 1;6 versus the recomputed value 1;7.

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[48] Istanbul Fatih 3418, folio 43v; Cairo DM 188/2, folio 15r; Leiden Or. 8 (1054), folio 35r; Berlin 5751, page 46. Another copy of the table for radius 7 occurs on folio 32r of the Leiden manuscript.

[49] It was the above-mentioned number of differences between cotangent and recomputation that gave me the idea of trying this method of computation. The probability that a cotangent value T_7' for radius 7 computed from the corresponding cotangent value T_{12} for radius 12 is different from a directly computed value T_7 can be calculated as follows (in all instances I assume the use of modern rounding):
For an arbitrary argument x, let $n+f$ with $n \in \mathbb{N}$ and $f \in [0,1\rangle$ denote the quotient $\frac{\mathrm{Sin}\,(90-x)}{\mathrm{Sin}\,x}$ expressed in units of the cotangent tables concerned (in the case of al-Baghdādī's cotangent tables for radii 12 and 7, the unit is a minute). First note that T_7' and T_7 cannot differ by more than one unit, since they are both whole numbers of units and we have

$$|T_7' - T_7| \le \left|T_7' - \tfrac{7}{12}T_{12}\right| + \tfrac{7}{12}\,|T_{12} - 12(n+f)| + |7(n+f) - T_7| \le \tfrac{1}{2} + \tfrac{7}{12}\cdot\tfrac{1}{2} + \tfrac{1}{2} = \tfrac{31}{24},$$

the second inequality holding since the error involved in modern rounding is half a unit at most. Now if f assumes a value f_0 from the set $\{\frac{1}{14}, \frac{3}{14}, \frac{5}{14}, \ldots, \frac{13}{14}\}$, we have $7\cdot(n+f) = m + \frac{1}{2}$ for a certain $m \in \mathbb{N}$. This implies that for f slightly smaller than f_0, T_7 will be equal to m, for f equal to or slightly larger than f_0, T_7 will be equal to $m+1$. Since $12\cdot(n+f_0)$ cannot be equal to $l+\frac{1}{2}$ for any $l \in \mathbb{N}$, it follows that T_{12} (and therefore also T_7') is constant in a neighbourhood of f_0. As a result T_7' will be either one unit larger than T_7 in an interval $[f_1, f_0\rangle$ or one unit smaller than T_7 in an interval $\langle f_0, f_1]$, where f_1 is such that $12\cdot(n+f_1) = l+\frac{1}{2}$ for a certain $l \in \mathbb{N}$. Proceeding thus we find that $T_7' - T_7$ equals $+1$ unit for $f \in \left[\frac{1}{24}, \frac{1}{14}\right\rangle \cup \left[\frac{5}{24}, \frac{3}{14}\right\rangle \cup \left[\frac{11}{24}, \frac{1}{2}\right\rangle \cup \left[\frac{5}{8}, \frac{9}{14}\right\rangle$, -1 unit for $f \in \left[\frac{5}{14}, \frac{3}{8}\right\rangle \cup \left[\frac{11}{14}, \frac{19}{24}\right\rangle \cup \left[\frac{13}{14}, \frac{23}{24}\right\rangle$, and 0 elsewhere. Assuming that f has a uniform distribution on the interval $[0,1\rangle$ (that this is a plausible assumption is shown in Section 1.2.4), we find that $T_7' - T_7$ has a probability of 0.095 of being equal to $+1$ unit and a probability of 0.054 of being equal to -1 unit. The cotangent table for radius of the base circle 7 in the Baghdādī Zīj has four errors more (17 versus 13) than we would expect from these probabilities. The proportion of positive and negative errors (11:6) is in good agreement with the expected proportion.

4.3.4 Solar Declination (folios 129$^{\mathrm{v}}$–130$^{\mathrm{v}}$, 1$^{\mathrm{st}}$ column)

Definition. The solar declination is the orthogonal distance on the sphere between the sun and the celestial equator. The solar declination is usually tabulated as a function of the true solar longitude λ:

$$\delta(\lambda) = \arcsin(\sin\lambda \cdot \sin\varepsilon), \tag{4.2}$$

where ε is the obliquity of the ecliptic. Because of the symmetry relations $\delta(180-\lambda) = \delta(\lambda)$ for $\lambda \in [0, 90]$ and $\delta(180+\lambda) = -\delta(\lambda)$ for $\lambda \in [0, 180]$, most solar declination tables have quadruple entries, i.e. they display values for a single quadrant only. In many cases the determination of the underlying value of the obliquity is straightforward, since we have $\delta(90) = \varepsilon$ for every value of ε. Exceptions are declination tables of which the tabular values have a smaller number of sexagesimal places than the underlying obliquity value.[50] From the solar declination one can compute the solar altitude h_s by a simple addition (we have $h_s(\lambda) = 90 - \phi + \delta(\lambda)$, where ϕ is the geographical latitude; see Section 4.3.6). Furthermore the solar declination can be used to compute the right ascension and the equation of daylight, possibly through an intermediate tabulation of the tangent of declination (see Sections 4.3.7, 4.3.8 and 4.3.10).

Description. In the Baghdādī Zīj the solar declination is tabulated in the first column of a table on folios 129$^{\mathrm{v}}$–130$^{\mathrm{v}}$ which is headed "Table of the solar declination, which is the distance from the ecliptic, and the lunar latitude, which is the distance from the zodiac, and the second declination". In this table the solar declination is given to three sexagesimal fractional digits, the lunar latitude and the second declination to two. We will see that al-Baghdādī computed the solar declination accurately for obliquity value $\varepsilon = 23;35$. Of only four errors the absolute value exceeds 10 sexagesimal thirds.

folios	arguments	unit	ε	N	n	μ	σ
129$^{\mathrm{v}}$–130$^{\mathrm{v}}$, 1°	$1, 2, \ldots, 90$ (4E)	0;0,0,1	23;35	90	69	$-2'''$	$6'''$

Analysis. From $T(90) = 23;35,0,0$ it follows that al-Baghdādī's declination table, like most of his tables, is based on obliquity of the ecliptic $\varepsilon = 23;35$. By comparing the manuscript values with a recomputation, we can easily correct seven large scribal errors (see Section 4.4.4 of the Apparatus). The error pattern of the solar declination table is irregular; the errors tend to be negative, are small for arguments close to 0° and increase in absolute value to approximately 10 thirds for arguments in the neighbourhood of 90°. I have not been able to explain the error pattern completely. The use of inverse linear interpolation in an accurate sine table with integer arguments leads to positive differences between solar declination and recomputation which become as large as 3 seconds. The use of inverse linear interpolation between accurate sine values for every 15′ leads to a regular difference pattern with a systematic deviation which deserves further investigation. Outliers in the declination values occur for arguments 29 ($-11'''$), 72 ($+13'''$) and 79 ($-39'''$). These outliers have *not* been excluded from the error statistics table.

[50] For example, I found that the declination table in Ptolemy's Handy Tables, which has values to minutes, is based on obliquity 23;51,20 (unpublished result). The table can be found in Stahlman 1959, p. 260. See also Section 3.2.2.2.

4.3.5 Second Declination (folios 129ᵛ–130ᵛ, 3rd column)

Definition. The second declination δ_2 is the arc of the great circle perpendicular to the ecliptic between the sun and the celestial equator. The second declination can be computed as

$$\delta_2(\lambda) = \arctan(\sin\lambda \cdot \tan\varepsilon), \tag{4.3}$$

where λ is the true solar longitude and ε is the obliquity of the ecliptic. The second declination satisfies the symmetry relations $\delta_2(180-\lambda) = \delta_2(\lambda)$ for $\lambda \in [0,90]$ and $\delta_2(180+\lambda) = -\delta_2(\lambda)$ for $\lambda \in [0,180]$ and is therefore usually tabulated for a single quadrant only. As for the solar declination we have $\delta_2(90) = \varepsilon$ for every value of ε, hence in most cases the determination of the underlying obliquity value is straightforward.

Description. In the Baghdādī Zīj the second declination is tabulated in the third column of a table on folios 129ᵛ–130ᵛ which is headed "Table of the solar declination, which is the distance from the ecliptic, and the lunar latitude, which is the distance from the zodiac, and the second declination". In this table the second declination and the lunar latitude are given to two sexagesimal fractional digits, the solar declination (see Section 4.3.4) to three. We will see that al-Baghdādī's table for the second declination is identical to the table in Kushyār ibn Labbān's Jāmiʿ Zīj apart from a number of strange differences that could be attributed to copying errors.

folios	arguments	unit	ε	N	n	μ	σ
129ᵛ–130ᵛ, 3°	$1, 2, \ldots, 90$ (4E)	0;0,1	23;35	90	57	$+59'''$	$2''$

Analysis. Al-Baghdādī's second declination table can also be found in four manuscripts of Kushyār ibn Labbān's Jāmiʿ Zīj.[51] By comparing al-Baghdādī's table with Kushyār's versions, we can correct four scribal errors (see Section 4.4.5 of the Apparatus). There remain eleven peculiar, apparently related differences: al-Baghdādī's values for arguments 50 to 59 are all $4''$ larger than Kushyār's values, al-Baghdādī's value for argument 60 is $4''$ smaller. It seems as if al-Baghdādī made use of tabular differences to "reconstruct" Kushyār's second declination table. In fact, all four copies of Kushyār's table include a column displaying tabular differences. In both the Istanbul and Cairo manuscripts we find an incorrect tabular difference for argument 50 (0;15,22 instead of 0;15,18) which could explain the error of $+4''$ in al-Baghdādī's value for argument 50 and hence, if tabular differences were used, in the subsequent values. When arriving at the end of the column in the manuscript, al-Baghdādī may have discovered his mistake by comparing his value for $\delta_2(60)$ with Kushyār's value and he may have continued correctly. In this scenario I have no convincing explanation for the negative error for argument 60.

From $T(90) = 23;35,0$ it follows that al-Baghdādī's second declination table, like most of the tables in the Baghdādī Zīj, is based on obliquity of the ecliptic $\varepsilon = 23;35$. Starting

[51] Istanbul Fatih 3418, folios 83ᵛ–84ʳ, 2nd col.; Cairo DM 188/2, folio 12ʳ; Leiden Or. 8 (1054), folios 74ᵛ–75ᵛ, 2nd col.; and Berlin 5751, pages 144–146, 2nd col. In the Istanbul, Leiden and Berlin manuscripts the second declination is tabulated together with the solar declination, in the Cairo manuscript it occurs on a page of its own. I verified that al-Baghdādī's lunar latitude, which is tabulated on the same pages as the solar declination and the second declination and which is based on a maximum latitude of $4°46'$, is not related to any of the various lunar latitude tables in the manuscripts of the Jāmiʿ Zīj.

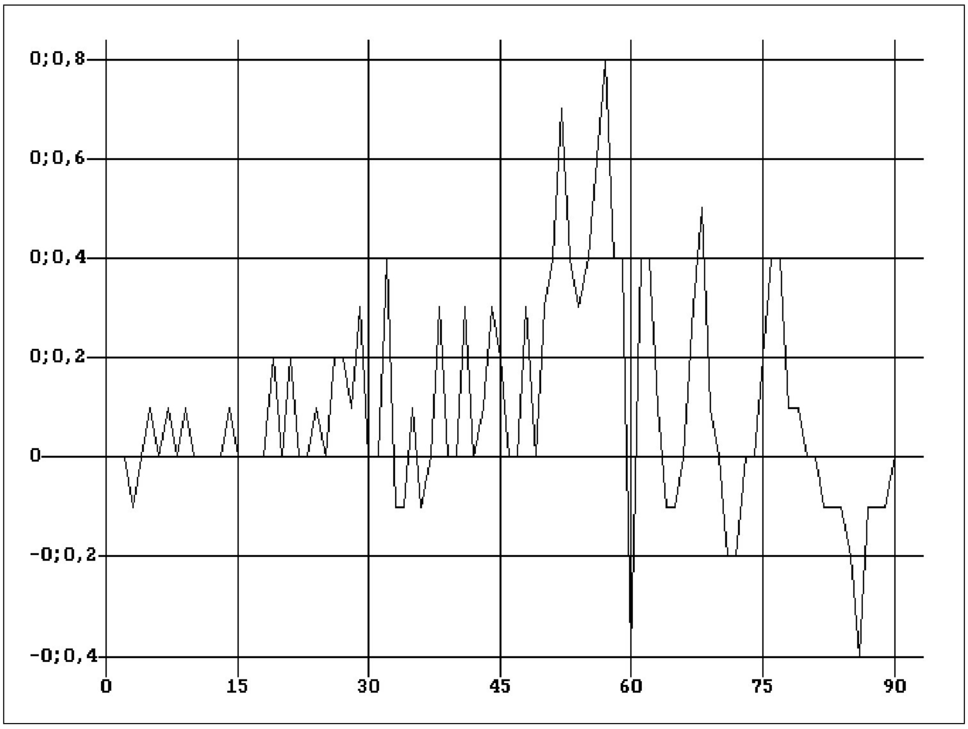

Figure 4.2: Errors in al-Baghdādī's second declination table

from argument 45 the errors occur in small clusters (see Figure 4.2). They are generally positive and show an increasing tendency for arguments between 1 and 60. For arguments between 60 and 90 the absolute value of the errors becomes somewhat smaller. There is a peculiar group of negative errors for arguments 82 to 89. I superficially investigated the possibility of linear or second order interpolation between accurately computed nodes, of a slightly different value for the obliquity of the ecliptic (for example as a result of the rounding of $\tan\varepsilon$), and of inverse linear interpolation in an accurate tangent table. None of these possibilities led to a better recomputation for al-Baghdādī's second declination.

4.3.6 Solar Altitude (folios 119$^{\text{v}}$–120$^{\text{r}}$)

Definition. The solar altitude $h_s(\lambda)$ is defined by

$$h_s(\lambda) = 90 - \phi + \delta(\lambda), \tag{4.4}$$

where the argument λ is the true solar longitude, ϕ is the geographical latitude for which the table is intended and $\delta(\lambda) \stackrel{\text{def}}{=} \arcsin(\sin\lambda \cdot \sin\varepsilon)$ is the solar declination (see section 4.3.4), which depends on the obliquity of the ecliptic ε. The solar altitude satisfies the symmetry relations $h_s(180-\lambda) = h_s(\lambda)$ for $\lambda \in [0, 90]$ and $h_s(180+\lambda) = 180-2\phi-h_s(\lambda)$ for $\lambda \in [0, 180]$. Therefore the solar altitude is usually tabulated with double entries and gives values for the first/second quadrants and for the third/fourth quadrants. Note that in most cases the parameters of a solar altitude table can be determined in a straightforward way, since $h_s(0) = 90-\phi$, $h_s(90) = 90-\phi+\varepsilon$ and $h_s(270) = 90-\phi-\varepsilon$. By subtracting $90-\phi$ from the tabular values, the underlying solar declination table can easily be reconstructed.

Description. The Baghdādī Zīj contains a table for the solar altitude on folios 119ᵛ–120ʳ, which is headed "Table of the maximum solar elevation at midday in Baghdad and wherever the latitude is 33°25′". We will see that this table was accurately computed from al-Baghdādī's solar declination table for the geographical latitude indicated. A second table for the solar altitude can be found in the fourth column of the table on folios 143ᵛ–149ʳ, which is attributed to al-Baghdādī himself and will be analysed in Section 4.3.14.4.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
119ᵛ–120ʳ	$1, 2, \ldots, 90,\ 271, 272, \ldots, 360$ (2E)	0;0,0,1	23;35	33;25	90	69	$-2'''$	$6'''$

Analysis. From $T(0) = 56;35,0,0$ and $T(90) = 80;10,0,0$ it follows that the table for the solar altitude on folios 119ᵛ–120ʳ is based on $\phi = 33;25$ and $\varepsilon = 23;35$. In fact, it can easily be verified that the table was computed directly from the declination table on folios 129ᵛ–130ᵛ (see Section 4.3.4). In 180 values we find only 12 differences between the solar altitude in the manuscript and a recomputation based on al-Baghdādī's declination table. Most of these differences can be explained as scribal errors, corrections of which are listed in Section 4.4.6 of the Apparatus.

4.3.7 Tangent of Declination (folio 235ʳ)

Definition. The tangent of the solar declination $t_\delta(\lambda)$ is given by

$$t_\delta(\lambda) = \mathrm{Tan}\,\delta(\lambda) = R \cdot \tan\delta(\lambda), \tag{4.5}$$

where the argument λ is the true solar longitude, R is the radius of the base circle for the trigonometric functions, and $\delta(\lambda) \stackrel{\mathrm{def}}{=} \arcsin(\sin\lambda \cdot \sin\varepsilon)$ is the solar declination (see section 4.3.4), which depends on the obliquity of the ecliptic ε. Because of the symmetry relations $t_\delta(180-\lambda) = t_\delta(\lambda)$ for $\lambda \in [0, 90]$ and $t_\delta(180+\lambda) = -t_\delta(\lambda)$ for $\lambda \in [0, 180]$, the tangent of declination was usually tabulated for the first quadrant only. Tables for the tangent of declination were used to compute the right ascension $\alpha(\lambda)$ (see Section 4.3.8) according to the formula

$$\alpha(\lambda) = \mathrm{arcSin}\left(\frac{R \cdot \mathrm{Tan}\,\delta(\lambda)}{\mathrm{Tan}\,\varepsilon}\right) \tag{4.6}$$

and the equation of daylight $\Delta(\lambda)$ according to

$$\Delta(\lambda) = \mathrm{arcSin}\,(\mathrm{Tan}\,\delta(\lambda) \cdot \mathrm{Tan}\,\phi / R). \tag{4.7}$$

The Baghdādī Zīj also contains a table for the sine of the equation of daylight (see Section 4.3.9), which can be computed from the tangent of declination by multiplying by the constant $\tan\phi$. We do not find the precise value of the obliquity of the ecliptic in a table for the tangent of declination. Instead we find for argument 90 a value for $\tan\varepsilon$ which may be subject to errors due to the computation of the tangent and to rounding.

Description. The Baghdādī Zīj contains a table called "Table of the parts of the ascension which is the tangent of the declination of every degree" on folio 235ʳ. We will see that this table is based on obliquity ε = 23;35, but I have not been able to explain the error pattern of the table. In Section 4.3.9 it will be shown that the tangent of declination table was used for the calculation of the table for the sine of the equation of daylight which occurs on folio 235^{v}.

folio	arguments	unit	R	ε	N	n	μ	σ
235^{r}	$1, 2, \ldots, 90$ (4E)	0;0,0,1	60	23;35	90	70	$-1'''$	$9'''$

Analysis. Since the table for the tangent of declination in the Baghdādī Zīj has $T(90) =$ 26;11,33,16, which is the exact value for Tan 23;35 to three sexagesimal fractional digits, we conclude that the table is based on radius of the base circle 60 and obliquity of the ecliptic 23;35. By comparing the manuscript values with recomputed values we find two obvious scribal errors for arguments 17 and 19 (see Section 4.4.7 of the Apparatus). The general error pattern is irregular, but the errors tend to become larger towards argument 90. There are many small outliers (of which the eight largest are listed in the Apparatus), which cannot be identified as scribal errors and are probably the result of the particular method according to which the values for the tangent of declination were calculated. I investigated various possible methods of computation, involving the use of al-Baghdādī's declination table and of linear interpolation in his tangent table. However, none of the investigated methods led to a better agreement than precise calculation. Note that the use of linear interpolation in any tangent table with an argument increment of 1° leads to positive errors as large as $8\frac{1}{2}$ seconds in the tangent of declination.[52]

4.3.8 Right Ascension (folios 141^{v}–143^{r})

Definition. The right ascension $\alpha(\lambda)$ can be defined as follows. Let Λ be the point on the ecliptic with longitude λ and let P be the orthogonal projection of Λ onto the equator. Then $\alpha(\lambda)$ is the length of the equatorial arc between the vernal point and P, measured in the direction of the solar motion. For $\lambda \in [0, 90\rangle$ we have

$$\alpha(\lambda) = \arctan(\cos\varepsilon \cdot \tan\lambda), \tag{4.8}$$

where ε as usual denotes the obliquity of the ecliptic. For $\lambda \in [90, 360]$ the right ascension follows from the symmetry relations $\alpha(180-\lambda) = 180-\alpha(\lambda)$ and $\alpha(180+\lambda) = 180+\alpha(\lambda)$. The right ascension was always tabulated for arguments in all four quadrants. Most mediaeval astronomers computed the right ascension according to the formula

$$\alpha(\lambda) = \mathrm{arcSin}\left(\frac{R \cdot \mathrm{Tan}\,\delta(\lambda)}{\mathrm{Tan}\,\varepsilon}\right), \tag{4.9}$$

using a tabulation for the solar declination $\delta(\lambda)$ (see Section 4.3.4) or for the tangent of declination $t_\delta(\lambda)$ (see Section 4.3.7). Although the declination also depends on the obliquity of the ecliptic, we need not take into account the possibility that two different values of

[52] I found that in the case of Kushyār's table for the tangent of declination in Fatih 3438, folio 84^{r}, the use of Kushyār's declination values and linear interpolation in his tangent table leads to a significantly better agreement than exact recomputation (unpublished result).

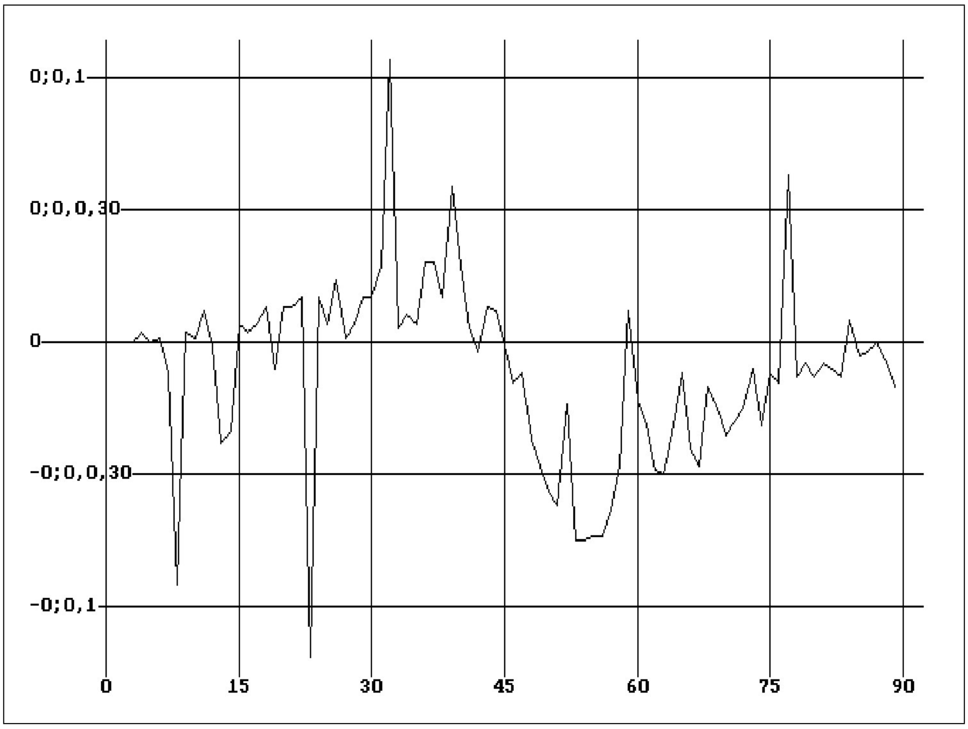

Figure 4.3: Errors in al-Baghdādī's right ascension table

this parameter are involved; even a very small difference between the two obliquity values leads to large anomalies in the right ascension which would have been unacceptable to any mediaeval astronomer.[53]

Description. On folios 141^v-143^r of the Baghdādī Zīj we find a table headed "Ascension of the signs at sphaera recta", which will turn out to be based on obliquity 23;35 and could have been computed from al-Baghdādī's table for the tangent of declination.

folios	arguments	unit	ε	N	n	μ	σ
141^v–143^r	$1, 2, \ldots, 360$	0;0,0,1	23;35	90	84	$-6'''$	$21'''$

Analysis. Even though the final sexagesimal digit of the right ascension values turns out to be inaccurate, there is no doubt that the underlying value of the obliquity of the ecliptic is ε = 23;35.[54] The error pattern for this value shows some peculiar trends (see Figure 4.3). I have not found sufficient evidence that al-Baghdādī's right ascension table was computed from the table for the tangent of declination on folio 235^r. The relatively large errors in the tangent of declination for arguments $8, 13, 23, 32$ and 39 are reflected in more or less proportional errors of the same sign in the right ascension values for these arguments. On the other hand, no correspondence between the errors in the two tables

[53]For instance, the use of obliquity values 23;51 and 23;51,20 leads to a value 88;41 for $\alpha(90)$ which differs by more than a degree from the correct value 90;0.

[54]Since the errors in the table seem to be correlated (see below), we cannot use all tabular values from the first quadrant for the estimation of the obliquity. If we use, for example, every fifth tabular value, we obtain a 95 % confidence interval $\langle$ 23;34,59,55 , 23;35,1,26 $\rangle$.

can be observed in the second half of the first quadrant. It is possible that in this part of the table the error made in the calculation of the arcsine in formula (4.9) overwhelms any error in the tangent of declination table. It can be checked that the use of al-Baghdādī's tangent of declination table and inverse linear interpolation between accurate sine values for integer arguments leads to negative errors in the right ascension with an absolute value as large as 2 minutes. Inverse linear interpolation in a sine table displaying values for every $15'$ seems to explain the errors in al-Baghdādī's right ascension for arguments between 1 and 45 quite well. However, for arguments between 45 and 90 a different method of computation must have been used.

4.3.9 Sine of the Equation of Daylight (folio 235$^{\text{v}}$)

Definition. The sine of the equation of daylight s_Δ is an auxiliary function which occurs together with the tangent of declination (see Section 4.3.7) at the end of the Baghdādī Zīj. The sine of the equation of daylight is given by

$$s_\Delta(\lambda) = t_\delta(\lambda) \cdot \tan\phi = R \cdot \tan\delta(\lambda) \cdot \tan\phi, \tag{4.10}$$

where ϕ is the geographical latitude and $t_\delta(\lambda) \stackrel{\text{def}}{=} \tan(\arcsin(\sin\lambda \cdot \sin\varepsilon))$ is the tangent of declination, which depends on the radius of the base circle R and on the obliquity of the ecliptic ε. The sine of the equation of daylight satisfies the symmetry relations $s_\Delta(180-\lambda) = s_\Delta(\lambda)$ and $s_\Delta(180+\lambda) = -s_\Delta(\lambda)$ for every λ. Therefore tables for the sine of the equation of daylight usually have quadruple entries. The latitude and obliquity values underlying a table for the sine of the equation of daylight are strongly correlated: if the latitude decreases by a small amount, the obliquity can be increased by a linearly correlated amount such that the resulting table for the sine of the equation of daylight is very close to the original table. Consequently, the 95 % confidence region obtained from a least squares estimation as described in Section 2.4 has a highly oblong shape and a negative slope, and therefore gives much more accurate information about the underlying parameter values than the marginal 95 % confidence intervals (i.e. the 95 % confidence intervals calculated for the separate parameters).

The equation of daylight, which is discussed in Section 4.3.10 below, can be computed from the sine of the equation of daylight by taking the arcsine, for instance by means of inverse interpolation in a sine table.

Description. The table for the sine of the equation of daylight in the Baghdādī Zīj occurs on folio 235$^{\text{v}}$ and is headed "Sine of the equation of daylight for Baghdad and wherever the latitude is 33°25′". It will be shown that this table was computed from the table for the tangent of declination immediately preceding it by multiplying the values from that table by a highly accurate value of $\tan\phi$.

folio	arguments	unit	R	ε	ϕ	N	n	μ	σ
235$^{\text{v}}$	$1, 2, \ldots, 90$ (4E)	0;0,0,1	60	23;35	33;25	90	68	$-1'''$	$7'''$

Analysis. By looking at the first differences of al-Baghdādī's table for the sine of the equation of daylight, we can locate four large scribal errors for arguments $7, 9, 47$ and 48;

the corrections are given in Section 4.4.9 of the Apparatus. Since $T(90) = 17;16,54,19$ we conclude that $R = 60$. A least squares estimation confirms the expected values $\varepsilon = 23;35$ for the obliquity of the ecliptic and $\phi = 33;25$ for the geographical latitude. The error pattern for these parameter values is similar to that of the table for the tangent of declination on the preceding page of the Baghdādī Zīj (see Section 4.3.7): relatively small errors are mixed with irregularly occurring larger ones. In fact, the errors in the sine of the equation of daylight and those in the tangent of declination have a significant correlation coefficient of +0.45. Therefore I compared the values for the sine of the equation of daylight values with a recomputation obtained by multiplying al-Baghdādī's values for the tangent of declination by a precise value for tan 33;25. The agreement turns out to be extremely good; after 10 more obvious scribal errors in the table for the sine of the equation of daylight have been corrected (see the Appendix), the difference statistics are $N = 90$, $n = 11$, $\mu = -10^{\text{iv}}$ and $\sigma = 3'''$.

It can be seen that the value for $\tan\phi$ used by al-Baghdādī must have been accurate to at least four sexagesimal places. For the values $\tan\phi = 0;39,35$ and $\tan\phi = 0;39,35,16$, mentioned in al-Baghdādī's numerical examples,[55] a recomputation from al-Baghdādī's table for the tangent of declination shows a systematic deviation from his table for the sine of the equation of daylight; the numbers of differences amount to 90 and 77 respectively. Even for $\tan\phi = 0;39,35,15,56$ we find 21 generally negative differences, significantly more than for the precise value of tan 33;25.

4.3.10 Equation of Daylight (folio 117$^{\text{r}}$)

Definition. The equation of daylight Δ (also called *ascensional difference*) is given by the modern formula:

$$\Delta(\lambda) = \arcsin(\tan\delta(\lambda)\cdot\tan\phi), \tag{4.11}$$

where ϕ is the geographical latitude and $\delta(\lambda) \stackrel{\text{def}}{=} \arcsin(\sin\lambda\cdot\sin\varepsilon)$ is the solar declination (see Section 4.3.4), which depends on the obliquity of the ecliptic ε. The equation of daylight is used in timekeeping to compute rising times (it is the difference between right and oblique ascension) and to compute the length of daylight and the length of a seasonal hour (see the following sections). The parameter values of a table for the equation of daylight are strongly correlated: if the obliquity decreases by a small amount, the latitude can be increased by a linearly correlated amount such that the resulting equation of daylight table is practically the same as the original table. Consequently, the 95 % confidence region obtained from a least squares estimation as described in Section 2.4 has a highly oblong shape, and therefore gives much more accurate information about the underlying parameter values than the marginal 95 % confidence intervals.

Description. Folio 117$^{\text{r}}$ of the Baghdādī Zīj displays a table called "Table of the equation of daylight for Baghdad and for latitude 33°25′". Below we will show that this table was computed from al-Baghdādī's table for the sine of the equation of daylight on folio 235$^{\text{v}}$

[55] The value 0;39,35 for $\tan\phi$ occurs in the example of the calculation of the equation of daylight on folio 233$^{\text{v}}$, lines 3 ff. The value 0;39,35,16 is used in the example for the oblique ascension on folio 230$^{\text{v}}$, lines 9 ff.

(see Section 4.3.9). To calculate the required arcsines al-Baghdādī performed inverse linear interpolation in a sine table with values accurate to four sexagesimal places for every 15′ of the argument. Since we have seen that Abu'l-Wafāʾ computed a sine table in this format, we have thus found more evidence that al-Baghdādī copied tables from this author (cf. Sections 4.3.1 and 4.3.3).

folio	arguments	unit	ε	ϕ	N	n	μ	σ
117r	$1, 2, \ldots, 90$ (4E)	0;0,0,1	23;35	33;25	90	80	$+2'''$	$10'''$

Analysis. By inspecting the first order differences of the table for the equation of daylight, we can correct seven large scribal errors (see Section 4.4.10 of the Apparatus). Four more scribal errors, also listed in the Apparatus, can be corrected by recomputing the oblique ascension table on folios 139v–141r using the table for the equation of daylight (this recomputation is explained in Section 4.3.13). A least squares estimation confirms that the equation of daylight was computed for the expected values 23;35 for the obliquity and 33;25 for the latitude, even though these values lie just outside the 95 % confidence region and the minimum obtainable standard deviation is as large as ten thirds.[56]

As in the table for the sine of the equation of daylight on folio 235v the errors in the equation of daylight are generally small, but show incidental outliers with an absolute value of $14'''$ or more. In most cases the outliers in the equation of daylight correspond to outliers with the same sign in the sine of the equation of daylight. The correlation coefficient +0.50 of the sets of errors of both tables also suggests that the tables are related. We will now investigate the precise method according to which the arcsine of the values for the sine of the equation of daylight was computed in order to obtain values for the equation of daylight.

First we note that no inverse linear interpolation in a sine table with accurate values for integer arguments was applied: the differences between al-Baghdādī's values for the equation of daylight and values thus recomputed have a standard deviation of $71'''$, whereas the differences between al-Baghdādī's values for the equation of daylight and exact values have a standard deviation of $10'''$ (see the error statistics table above).

I compared al-Baghdādī's table for the equation of daylight with recomputations from his table for the sine of the equation of daylight based on inverse linear interpolation in sine tables having values accurate to four sexagesimal places and argument increments $3, 5, 6, 7\frac{1}{2}, 10, 12, 15, 20$ and 30 minutes. It turned out that the agreement is very good for argument increment 15′, but not for any of the other increments.[57] It can be checked that the agreement is significantly poorer if the sine values are accurate to only three

[56] Marginal 95 % confidence intervals are ⟨23;34,59,45 , 23;35,2,39⟩ for the obliquity of the ecliptic and ⟨33;24,56,55 , 33;25,0,24⟩ for the geographical latitude.

[57] For argument increment 15′ there are 16 differences between al-Baghdādī's table for the equation of daylight and the recomputation: eleven differences of $-1'''$, $+1'''$ or $-2''$, and five outliers for arguments 31 (difference $-59''$), 52 ($+5''$), 56 ($-18'''$), 62 ($+5'''$) and 68 ($+46'''$). Disregarding the five outliers the difference statistics are $N = 85$, $n = 11$, $\mu = -3^{\text{iv}}$, $\sigma = 24^{\text{iv}}$. Of the other argument increments the "best" difference statistics are those for 12′: $N = 85$, $n = 66$, $\mu = \frac{3}{4}'''$, $\sigma = 2\frac{1}{2}'''$. The explanation for the fact that one of the recomputations is so much better than all the others is as follows. The error in a tabular value calculated by means of interpolation depends on the following:

sexagesimal places. Also if the sine values are accurate to five sexagesimal places, we find more differences between manuscript and recomputation.[58] We conclude that al-Baghdādī computed the equation of daylight from his table for the sine of the equation of daylight by calculating the required arcsines by means of inverse linear interpolation in an accurate sine table having four sexagesimal places and argument increment 15 minutes.

In Section 4.3.1 we have seen that both al-Bīrūnī and Abu'l-Wafāʾ computed accurate sine tables with values to four sexagesimal places for every $15'$ of the argument. The possibility suggested in that section that al-Baghdādī took the values of his sine table from Abu'l-Wafāʾ is supported by the present result: although al-Baghdādī only included sine values for integer arguments in his zīj, his table for the equation of daylight can be shown to rely on a sine table having argument increment $15'$.

4.3.11 Length of Daylight (folios 117$^{\text{v}}$–118$^{\text{r}}$)

Definition. The length of daylight L gives the number of equal hours between sunrise and sunset as a function of the true solar longitude λ. The length of daylight is a linear function of the equation of daylight Δ (see Section 4.3.10):

$$L(\lambda) = \frac{90 + \Delta(\lambda)}{7\frac{1}{2}} = \frac{90 + \arcsin(\tan\delta(\lambda) \cdot \tan\phi)}{7\frac{1}{2}}, \tag{4.12}$$

where ϕ is the geographical latitude and $\delta(\lambda) \stackrel{\text{def}}{=} \arcsin(\sin\lambda \cdot \sin\varepsilon)$ is the solar declination (see Section 4.3.4), which depends on the obliquity of the ecliptic ε. The length of daylight satisfies the symmetry relations $L(180 - \lambda) = L(\lambda)$ and $L(180 + \lambda) = 24 - L(\lambda)$ for every λ. Therefore it is usually tabulated with double entries and gives values both for the zodiacal signs Aries, Taurus and Gemini and for the signs Capricornus, Aquarius and Pisces.

Description. The Baghdādī Zīj contains a table for the length of daylight on folios 117$^{\text{v}}$–118$^{\text{r}}$. This table is headed "Table of [the length of daylight in] equal hours for Baghdad and wherever the latitude is $33°25'$". Below we will see that the same table can be found in the Berlin version of the zīj of Ḥabash al-Ḥāsib. It can be shown that the table was computed from the equation of daylight table on the preceding folio of the Baghdādī Zīj, which does *not* occur in Ḥabash's zīj. The rounding involved in the computation was performed in the modern way.

- How well can the tabulated function be approximated by the type of interpolation concerned? For instance, linear interpolation in a table for a strongly curved function leads to large errors.
- How far is the tabular value to be calculated by means of interpolation removed from the nodes (i.e. the exactly calculated tabular values)? Close to the nodes the interpolation introduces a negligible error; half-way between two nodes the errors will generally be largest.

We thus see that for a given function the error pattern of a table computed by means of interpolation is highly dependent on the choice of the nodes. In fact, the error pattern turns out to be a kind of fingerprint which matches one particular set of nodes nicely, but not other sets.

[58] If the five outliers mentioned in footnote 57 are disregarded, the difference statistics for the case of inverse linear interpolation in a sine table having accurate values to five sexagesimal places and argument increment $15'$ are $N = 85$, $n = 21$, $\mu = -6'''$, $\sigma = 32^{\text{iv}}$.

Another table for the length of daylight occurs in the second column of a table on folios 143^v–149^r, which is attributed to al-Baghdādī and will be analysed in Section 4.3.14.2.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
117^v–118^r	$1, 2, \ldots, 90, 271, 272, \ldots, 360$ (2E)	0;0,0,1	23;35	33;25	90	40	-11^{iv}	$7'''$

Analysis. A second copy of al-Baghdādī's table for the length of daylight on folios 117^v–118^r can be found on folios 112^r–112^v of the manuscript Berlin 5750 of the zīj of Ḥabash al-Ḥāsib. By assuming that al-Baghdādī's table satisfies the symmetry relation $T(-\lambda) = 24 - T(\lambda)$ and by comparing his values with those in Ḥabash's zīj, we can correct ten scribal errors (see Section 4.4.11 of the Apparatus). As we would expect, al-Baghdādī's table for the length of daylight was computed from his table for the equation of daylight on folio 117^r by applying the linear relationship $L(\lambda) = (90 + \Delta(\lambda))\big/7\frac{1}{2}$ and modern rounding. For arguments between 1 and 90 there are only eight differences between al-Baghdādī's table and a table thus recomputed. Below (Section 4.3.12) we will see that al-Baghdādī's hour length table on the following folios was also computed from his table for the equation of daylight. I found that in seven cases al-Baghdādī did not use the values for the equation of daylight found in his zīj, but instead made use of different values of unknown origin.

4.3.12 Hour Length (folios 118^v–119^r)

Definition. The hour length H gives the length of a seasonal hour expressed in equatorial degrees as a function of the true solar longitude λ. The hour length is a linear function of the equation of daylight Δ (see Section 4.3.10):

$$H(\lambda) = \frac{90 + \Delta(\lambda)}{6} = \frac{90 + \arcsin(\tan\delta(\lambda) \cdot \tan\phi)}{6}, \tag{4.13}$$

where ϕ is the geographical latitude and $\delta(\lambda) \stackrel{\text{def}}{=} \arcsin(\sin\lambda \cdot \sin\varepsilon)$ is the solar declination (see Section 4.3.4), which depends on the obliquity of the ecliptic ε. The hour length satisfies the symmetry relations $H(180 - \lambda) = H(\lambda)$ and $H(180 + \lambda) = 30 - H(\lambda)$. Therefore the function is usually tabulated with double entries and gives values both for the zodiacal signs Aries, Taurus and Gemini and for the signs Capricorn, Aquarius and Pisces.

Description. On folios 118^v–119^r of the Baghdādī Zīj we find a table headed "Table of the parts of the hours for the City of Peace [i.e. Baghdad] and whenever the latitude is 33°25′". Below we will see that the same tabular values can be found in a hour length table in the Berlin version of the zīj of Ḥabash al-Ḥāsib. It can be shown that the table was computed from the table for the equation of daylight on folio 117^r of the Baghdādī Zīj, which does *not* occur in Ḥabash's zīj. The rounding involved in the computation was performed in the modern way.

Another hour length table is found in the third column of the table on folios 143^v–149^r, which is attributed to al-Baghdādī and will be analysed in Section 4.3.14.3.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
118^{v}–119^{r}	$1, 2, \ldots, 90, 271, 272, \ldots, 360$ (2E)	0;0,0,1	23;35	33;25	90	48	$+45^{iv}$	$3'''$

Analysis. The tabular values for arguments 1 to 89 of al-Baghdādī's table for the hour length on folios 118^{v}–119^{r} can also be found in a hour length table on folios 109^{r}–111^{v} of the manuscript Berlin 5750 of the zīj of Ḥabash al-Ḥāsib. In that table al-Baghdādī's values occur for arguments 1 to 60 and 91 to 120. I found that Ḥabash's other tabular values, which are given to seconds only, are based on obliquity $\varepsilon = 23;51$ and latitude $\phi = 33;0$.[59] By assuming that al-Baghdādī's table satisfies the symmetry relation $T(-\lambda) = 30 - T(\lambda)$ and by comparing his values with those in Ḥabash's zīj, we can correct 13 scribal errors (see Section 4.4.11 of the Apparatus). As we would expect, al-Baghdādī's table for the hour length was computed from his table for the equation of daylight on folio 117^{r} by applying the linear relationship $H(\lambda) = (90 + \Delta(\lambda))/\,6$ and modern rounding. Indeed for arguments between 1 and 90 there are only six differences between al-Baghdādī's table and a table thus recomputed. In Section 4.3.11 above it was shown that al-Baghdādī's table for the length of daylight was also computed from his table for the equation of daylight.

4.3.13 Oblique Ascension

Definition. The oblique ascension $\rho(\lambda)$ for a given terrestrial latitude can be defined as follows. Let Λ be the point on the ecliptic with longitude λ and let P be the point on the equator which rises simultaneously with Λ. Then $\rho(\lambda)$ is the length of the equatorial arc between the vernal point and P, measured in the direction of the solar motion. $\rho(\lambda)$ can be computed according to

$$\rho(\lambda) = \alpha(\lambda) - \Delta(\lambda), \tag{4.14}$$

where $\alpha(\lambda)$ is the right ascension, given by formula (4.8) for $\lambda \in [0, 90\rangle$ and by the symmetry relations $\alpha(180 - \lambda) = 180 - \alpha(\lambda)$ and $\alpha(180 + \lambda) = 180 + \alpha(\lambda)$ for $\lambda \in [90, 360]$, and $\Delta(\lambda)$ is the equation of daylight, which is given by formula (4.11) and satisfies the symmetry relations $\Delta(180 - \lambda) = \Delta(\lambda)$ and $\Delta(180 + \lambda) = -\Delta(\lambda)$. As a result of the symmetry of the right ascension and the equation of daylight, the oblique ascension satisfies the symmetry relation $\rho(360 - \lambda) = 360 - \rho(\lambda)$ for every λ. Furthermore, the right ascension and the equation of daylight used for the calculation of a given oblique ascension table can be "extracted" by means of the identities

$$\alpha(\lambda) = 90 + \tfrac{1}{2}\left(\rho(\lambda) - \rho(180 - \lambda)\right) \tag{4.15}$$

and

$$\Delta(\lambda) = 90 - \tfrac{1}{2}\left(\rho(\lambda) + \rho(180 - \lambda)\right) \tag{4.16}$$

respectively, which hold for all $\lambda \in [0, 180]$.

The oblique ascension depends on the obliquity of the ecliptic ε and on the geographical latitude ϕ. Since both the right ascension and the equation of daylight depend on

[59] Unpublished result.

the obliquity, it is possible that two different values of ε are involved in a single table for the oblique ascension.[60] Therefore I will treat the oblique ascension as a function based on three different parameters. In the error statistics tables the obliquity value underlying the right ascension will be indicated by ε_α, the obliquity value underlying the equation of daylight, by ε_Δ. As in the case of a separate table for the equation of daylight, the obliquity of the ecliptic and the geographical latitude underlying the equation of daylight used for the calculation of an oblique ascension table are strongly correlated. Therefore 95 % confidence regions for these two parameters obtained from a least squares estimation as described in Section 2.4 have highly oblong shapes and give much more accurate information about the underlying parameter values than the marginal 95 % confidence intervals.

General description. The Baghdādī Zīj contains no less than twelve tables for the oblique ascension: a table to sexagesimal thirds for the geographical latitude of Baghdad ($\phi = 33°25'$), a table to seconds for latitude $32°20'$, which occurs in a table attributed to al-Baghdādī himself analysed in Section 4.3.14.1, and a set of ten tables to minutes for latitudes $30°, 31°, 33°, 34°, \ldots, 40°$. Together these oblique ascension tables cover a large part of the range of latitude values in which mediaeval Islamic astronomers were interested.

4.3.13.1 The oblique ascension table for Baghdad (folios 139^v–141^r)

Description. The table on folios 139^v–141^r is the most accurate oblique ascension table in the Baghdādī Zīj. The table is headed "Ascensions of the signs in Baghdad, latitude 33°25′". We will see that it was computed by subtracting the equation of daylight values found on folio 117^r from the right ascension values on folios 141^v–143^r. Since all three tables involved have values to sexagesimal thirds, no rounding was necessary in the calculation of the oblique ascension.

folios	arguments	unit	ε_α	ε_Δ	ϕ	N	n	μ	σ
139^v–141^r	$1, 2, \ldots, 360$	0;0,0,1	23;35	23;35	33;25	180	172	$-2'''$	$23'''$

Analysis. It can easily be checked that the oblique ascension table for Baghdad on folios 139^v–141^r of the Baghdādī Zīj was computed by subtracting the equation of daylight table on folio 117^r from the right ascension table on folios 141^v–143^r. Values recomputed from these two tables differ from al-Baghdādī's oblique ascension values in only 80 out of 360 cases. Most of the differences can be explained from scribal errors in one of the three tables involved. By comparing al-Baghdādī's right ascension and equation of daylight tables with tables extracted from the oblique ascension by means of formulae (4.15) and (4.16), scribal errors in all three tables could be corrected (see Sections 4.4.8, 4.4.10 and 4.4.13 of the Apparatus). By assuming the symmetry $T(360-\lambda) = 360 - T(\lambda)$ for the oblique ascension table, nine more scribal errors could be retrieved.

[60] An example of an oblique ascension table based on two different values of the obliquity of the ecliptic is mentioned in Section 2.6.1.

4.3.13.2 The set of oblique ascension tables on folios 150^{v}–160^{r}

Description. On folios 150^{v}–160^{r} of the Baghdādī Zīj we find a set of oblique ascension tables to minutes for latitudes $30°, 31°, 33°, 34°, \ldots, 40°$. The headings of the tables mention both the latitude for which they are intended and the length of the longest day in hours. We will see below that the tables for latitudes 30° to 35° and for 37° were accurately computed on the basis of obliquity of the ecliptic $\varepsilon = 23;35$ and the indicated values for the geographical latitude. The tables for 38° to 40° were computed from an accurate right ascension table and from values for the equation of daylight determined by means of linear interpolation between accurate values for the strange set of arguments $\{1, 7, 13, \ldots, 85, 90\}$. The table for latitude 36°, finally, was computed from right ascension and equation of daylight values to minutes obtained by means of linear interpolation within intervals of 6°. Both the oblique ascension table for 36° itself and the two underlying tables can be found in the Jāmiʿ Zīj of Kushyār ibn Labbān. It seems plausible that al-Baghdādī composed the set of oblique ascension tables by taking a couple of tables from earlier sources and computing some others himself.

folios	arguments	unit	ε_α	n_α	ε_Δ	ϕ	n_Δ	N	n	μ	σ
150^{v}–151^{r}	$1, 2, \ldots, 360$	0;1	23;35	1	23;35	30	2	180	8	0	$13''$
151^{v}–152^{r}	$1, 2, \ldots, 360$	0;1	23;35	4	23;35	31	1	180	7	$-1''$	$12''$
152^{v}–153^{r}	$1, 2, \ldots, 360$	0;1	23;35	3	23;35	33	4	180	11	$-20'''$	$15''$
153^{v}–154^{r}	$1, 2, \ldots, 360$	0;1	23;35	1	23;35	34	3	180	10	$+1''$	$14''$
154^{v}–155^{r}	$1, 2, \ldots, 360$	0;1	23;35	2	23;35	35	1	180	5	$-20'''$	$10''$
155^{v}–156^{r}	$1, 2, \ldots, 360$	0;1	23;35	32	23;35	36	43	180	98	$+20''$	$58''$
156^{v}–157^{r}	$1, 2, \ldots, 360$	0;1	23;35	2	23;35	37	4	180	16	0	$18''$
157^{v}–158^{r}	$1, 2, \ldots, 360$	0;1	23;35	7	23;35	38	52	180	100	$+38''$	$56''$
158^{v}–159^{r}	$1, 2, \ldots, 360$	0;1	23;35	1	23;35	39	40	180	87	$+34''$	$54''$
159^{v}–160^{r}	$1, 2, \ldots, 360$	0;1	23;35	4	23;35	40	42	180	87	$+34''$	$54''$

Analysis. To all tables of this set I applied the following analysis techniques for oblique ascension tables:

1° Check for the use of (linear) interpolation by investigating the (first order) tabular differences.

2° Check the symmetry of the table and, if possible, correct deviations from that symmetry which can be explained as scribal errors.

3° Extract the underlying tables for the right ascension and the equation of daylight according to formulae (4.15) and (4.16). Note that, since the oblique ascension values are given to minutes, all extracted values are bound to have a number of seconds equal to 0 or 30. If the tables for the right ascension and the equation of daylight used for the computation of the oblique ascension both had values to minutes, then all extracted values would have zeros in the seconds' place; in other cases the digits 0 and 30 would occur approximately equally often. If the oblique ascension values are correct, it follows from formulae (4.15) and (4.16) that the extracted values contain errors of 30″ at most.

4° Estimate the underlying parameters of the extracted tables. It will turn out that for some tables the errors in the extracted values are correlated. In such cases I performed the parameter estimations for subsets of the tabular values for which the errors are independent, for example $\{1, 3, 5, \ldots, 89\}$ and $\{2, 4, 6, \ldots, 90\}$.

5° Determine the number of differences larger than 30″ between the extracted tables and recomputed values. The results (indicated by n_α for the right ascension and by n_Δ for the equation of daylight) are given in the error statistics table.

6° Recompute the oblique ascension on the basis of the parameter values found. As usual the error statistics are given in the table.

By applying these analysis techniques we find that the set of oblique ascension tables on folios 150ᵛ–160ʳ can be divided into three groups as far as accuracy and methods of computation are concerned. I will now discuss each of these groups.

Latitudes 30°, 31°, 33°, 34°, 35° and 37°. The oblique ascension tables for these latitude values and the tables for the right ascension and the equation of daylight that can be extracted from them are all very accurate. The tables are based on the parameter values that we expect: obliquity of the ecliptic 23;35 and the latitude values mentioned in the headings. The tables have very few deviations from the symmetry $\rho(360-\lambda) = 360-\rho(\lambda)$. The larger deviations can be corrected by inspecting the first order tabular differences or by comparing the tabular values with recomputed values. All corrections are given in Section 4.4.13.2 of the Apparatus. The deviations of $\pm 1'$ can often be corrected in such a way that the numbers of errors in the extracted tables and in the oblique ascension table itself become smaller. However, since these corrections cannot be confirmed by other means, I have not incorporated them.

For all oblique ascension tables in this group the digits 0 and 30 occur approximately equally often in the seconds' place of the extracted right ascension and equation of daylight. It follows that both underlying tables had values to seconds at least. However, since the oblique ascension values are generally accurate, it is impossible to determine dependencies on other tables, in particular on the accurate right ascension table on folios 141ᵛ–143ʳ of the Baghdādī Zīj and on the tangent of declination table on folio 235ʳ, from which the underlying equation of daylight could have been computed.

Latitudes 38°, 39° and 40°. Like the tables of the first group, the oblique ascension tables for latitudes 38° to 40° have very few deviations from the symmetry $\rho(360-\lambda) = 360-\rho(\lambda)$. Again we can correct the larger deviations by inspecting the tabular differences or by comparing the tabular values with recomputed values; the smaller ones have to remain uncorrected. The tabular differences of the oblique ascension tables in this group do not show patterns that point to the use of interpolation.

In each case the extracted right ascension turns out to be based on obliquity 23;35 and differs in very few cases by more than 30″ from recomputed values. The first order differences of the extracted equation of daylight exhibits clear jumps for arguments $6k+1$, $k = 0, 1, 2, \ldots, 14$, in between which the differences are almost constant. A sample of this

λ	$T_\Delta(\lambda{+}1)-T_\Delta(\lambda)$	λ	$T_\Delta(\lambda{+}1)-T_\Delta(\lambda)$	λ	$T_\Delta(\lambda{+}1)-T_\Delta(\lambda)$
45	0;15,30	60	0;12,30	75	0; 6, 0
46	0;15, 0	61	0;10,30	76	0; 6, 0
47	0;15,30	62	0;11, 0	77	0; 6,30
48	0;15, 0	63	0;10,30	78	0; 5,30
49	0;14, 0	64	0;10,30	79	0; 4, 0
50	0;14, 0	65	0;10,30	80	0; 3,30
51	0;14, 0	66	0;11, 0	81	0; 3,30
52	0;14, 0	67	0; 8,30	82	0; 4, 0
53	0;14, 0	68	0; 8, 0	83	0; 3,30
54	0;14, 0	69	0; 8,30	84	0; 3,30
55	0;12,30	70	0; 9, 0	85	0; 1, 0
56	0;12,30	71	0; 8,30	86	0; 1, 0
57	0;12,30	72	0; 8, 0	87	0; 1, 0
58	0;12, 0	73	0; 6, 0	88	0; 1,30
59	0;12,30	74	0; 6,30	89	0; 0,30

Table 4.1: Tabular differences of the extracted equation of daylight for latitude 38°

pattern (which is less obvious in the almost linear part of the equation of daylight close to argument 0) is shown in Table 4.1 (for each λ, $T_\Delta(\lambda)$ denotes the extracted value of the equation of daylight for argument λ). We conclude that the underlying equation of daylight for the tables in this group was computed by means of linear interpolation between nodes for arguments $6k+1$, $k = 0, 1, 2, \ldots, 14$ and (probably) 90. The nodes are based on ε = 23;35 and on the latitude values mentioned in the headings of the respective tables. In fact, for each table only one of the nodes differs by more than 30″ from recomputed values for these parameter values. If we recompute the underlying tables for the equation of daylight by applying linear interpolation between accurate nodes to seconds, we find only 5 differences larger than 30″ between the 90 extracted values and the recomputation for each table. I do not know why al-Baghdādī or his source performed the linear interpolation between arguments $6k+1$ (integer k) instead of between the more common arguments $6k$.

The error patterns of the oblique ascension tables in this group confirm that only the underlying equation of daylight was computed by means of linear interpolation. The patterns show a sine wave of period 360° with a maximum around argument 90, a minimum around argument 270, and zeros around arguments 0 and 180. If only the underlying right ascension had been computed by means of linear interpolation, the error pattern of the oblique ascension would show a sine wave of period 180° with maxima around arguments 45 and 225, minima around 135 and 315, and zeros around $0, 90, 180$ and 270. Finally, if both underlying tables or the oblique ascension values themselves had been calculated

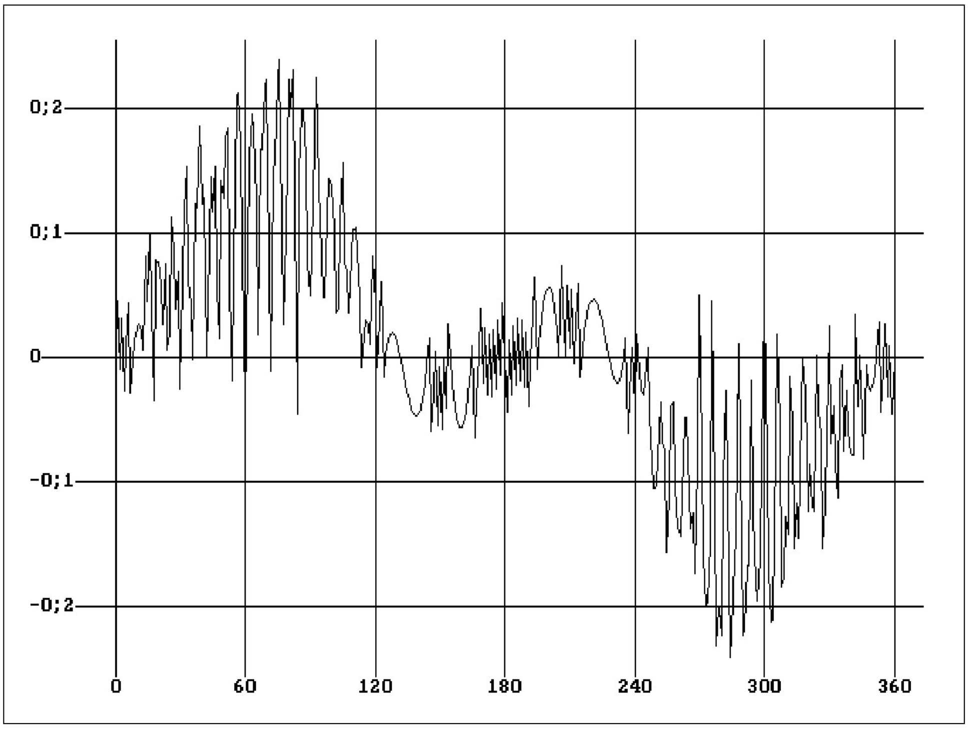

Figure 4.4: Errors in an oblique ascension table computed by means of linear interpolation

by means of linear interpolation, the resulting table would combine the errors in both underlying functions to produce a pattern like the one in Figure 4.4.[61]

Latitude 36°. The oblique ascension table for latitude 36° turns out to be different from the other tables in the set in various respects. The right ascension and equation of daylight that can be extracted from this table have a digit 0 in the seconds' place in 83 out of 90 values. We can conclude that both underlying tables had values to minutes and can be reconstructed precisely. The digits 30 in the seconds' place of seven extracted values turn out to be caused by deviations from the symmetry of the oblique ascension of size $\pm 1'$ and disappear when these deviations are corrected (see Section 4.4.13.2 of the Apparatus).

The first order differences of both the underlying right ascension and the equation of daylight reveal that linear interpolation between accurate values for multiples of 6 degrees was used for the computation of these tables.[62] In the case of the equation of daylight all but one of the nodes are correct for $\varepsilon = 23;35$ and $\phi = 36$. In the case of the right ascension, there are three negative errors and one positive error in the 15 nodes for $\varepsilon = 23;35$, but no other plausible value of the obliquity can be found which gives a better

[61] Figure 4.4 displays the differences between oblique ascension values to minutes calculated by means of linear interpolation within intervals of 6 degrees and exact values. The small curves that are visible around arguments $135, 160, 200$ and 225 point to (local) dependence of the tabular errors; cf. Section 1.2.4.

[62] This is confirmed by the error pattern of the oblique ascension table, which is of the type shown in Figure 4.4. It can be seen that for both tables exact linear interpolation was applied. In general, digits 30 in the seconds' place of internodal values were truncated.

agreement.

It turns out that al-Baghdādī's oblique ascension table for latitude 36° and the underlying right ascension and equation of daylight tables can be found in the Jāmiʿ Zīj of Kushyār ibn Labbān. The manuscript Istanbul Fatih 3418 contains all three tables,[63] the incomplete Cairo DM 188/2 only the right ascension.[64] The main set of tables in Berlin Ahlwardt 5751 and Leiden Or. 8 (1054) include an oblique ascension table for a different geographical latitude and a more accurate right ascension table.[65] However, among the appended tables in the Leiden manuscript we find both the right ascension and the equation of daylight tables underlying al-Baghdādī's oblique ascension table for latitude 36°.[66] Comparison with the oblique ascension table in the Istanbul manuscript confirms all corrections of deviations from the symmetry of al-Baghdādī's table given in the Apparatus.

4.3.14 The table attributed to al-Baghdādī on folios 143ᵛ–149ʳ

Description. On folios 143ᵛ–149ʳ of the Baghdādī Zīj we find a table headed "The oblique ascension, equal and seasonal hours and solar elevation for latitude 32;20 [in the] version of Abū al-Qāsim ibn Maḥfūẓ" [i.e. al-Baghdādī]. Although the title is somewhat ambiguous, it seems plausible that the table was computed by al-Baghdādī himself. The table consists of four columns displaying the functions mentioned in the heading, and covers twelve pages of the manuscript, one for every sign of the ecliptic. The first two pages of the table are displayed in Plate 4.1 on page 154. In Section 4.2 I mentioned that the value 32°20′ could be the geographical latitude of Wāsiṭ in Iraq.

Below it will be shown that the oblique ascension in the first column of the table is indeed based on the value 32;20 for the geographical latitude and on obliquity of the ecliptic 23;35. The oblique ascension was computed from a right ascension table of which almost half of the tabular values contain an error of $\pm 1''$; hence this table was less accurate than al-Baghdādī's own right ascension table. Furthermore, the oblique ascension was based on inaccurate equation of daylight values to minutes (!) which were computed by means of linear interpolation. Neither of the underlying tables can be found in the Baghdādī Zīj itself or in one of the extant early zījes that I inspected.

The second column of the table on folios 143ᵛ–149ʳ displays the length of daylight, the third column, the length of the hour. It can be shown that both columns were computed from the same equation of daylight values from which the oblique ascension in the first column was computed. The fourth column displays the solar altitude, which is based on the geographical latitude value 32;20 as well and which was probably computed from al-Baghdādī's solar declination table.

63 See Istanbul Fatih 3418, folios 84ᵛ (right ascension), 85ʳ (oblique ascension) and 85ᵛ (equation of daylight).

64 Cairo DM 188/2, folio 12ᵛ.

65 See Berlin Ahlwardt 5751, pages 148–152 and Leiden Or. 8 (1054), folios 76ᵛ–78ᵛ.

66 Leiden Or. 8 (1054), folios 134ʳ (right ascension) and 143ʳ (equation of daylight).

λ	$T_\Delta(\lambda)$	error	λ	$T_\Delta(\lambda)$	error	λ	$T_\Delta(\lambda)$	error
1	0;15		31	7;39	−1	61	13;41	
2	0;30		32	7;53	−1	62	13;50	
3	0;45	−1	33	8; 7		63	13;59	
4	1; 0	−1	34	8;21		64	14; 8	+1
5	1;15	−1	35	8;35		65	14;16	+1
6	1;30	−1	36	8;49		66	14;24	
7	1;45	−1	37	9; 2		67	14;32	+1
8	2; 0	−1	38	9;15		68	14;40	+1
9	2;15	−1	39	9;28	−1	69	14;48	+2
10	2;30	−2	40	9;41	−1	70	14;54	+1
11	2;45	−2	41	9;54	−1	71	15; 0	+1
12	3; 0	−2	42	10; 7	−1	72	15; 6	
13	3;15	−2	43	10;20	−1	73	15;12	
14	3;30	−2	44	10;33		74	15;18	+1
15	3;45	−2	45	10;45	−1	75	15;23	
16	4; 0	−2	46	10;57	−1	76	15;28	
17	4;15	−2	47	11; 9	−1	77	15;33	
18	4;30	−1	48	11;21	−1	78	15;37	
19	4;45	−1	49	11;33	−1	79	15;41	
20	5; 0	−1	50	11;45	−1	80	15;45	
21	5;15	−1	51	11;57		81	15;49	+1
22	5;30		52	12; 9	+1	82	15;52	+1
23	5;45		53	12;20	+1	83	15;54	
24	6; 0		54	12;31	+1	84	15;56	
25	6;15	+1	55	12;42	+1	85	15;58	
26	6;29	+1	56	12;52		86	16; 0	
27	6;43		57	13; 2		87	16; 1	
28	6;57		58	13;12		88	16; 2	
29	7;11		59	13;22		89	16; 2	
30	7;25	−1	60	13;32	+1	90	16; 2	

Table 4.2: The equation of daylight underlying al-Baghdādī's table on folios 143^{v}–149^{r}

4.3.14.1 Oblique Ascension (1st column)

Definition. The oblique ascension $\rho(\lambda)$ is defined in Section 4.3.13.

folios	arguments	unit	ε_α	ε_Δ	ϕ	N	n	μ	σ
143v–149r, 1°	$1, 2, \ldots, 360$	0;0,1	23;35	23;35	32;20	180	177	$+12''$	$51''$

Analysis. The oblique ascension column of the table on folios 143v–149r has only three large deviations from the symmetry $\rho(360-\lambda) = 360-\rho(\lambda)$, namely for arguments 16/344, 36/324 and 141/219, and five deviations of $\pm 1''$. If we extract the equation of daylight used for the computation of this oblique ascension table by means of formula (4.16), we find that 68 of the 90 extracted values have a whole number of minutes, although the oblique ascension values are given to seconds.[67] Nineteen values differ by only $30'''$ from a whole number of minutes, the remaining three differ by $2''$ (argument 16) or $30''$ (arguments 39 and 88). We conclude that the underlying table for the equation of daylight had values calculated only to minutes. The above-mentioned deviations from the symmetry of the oblique ascension can be corrected in such a way that seven more extracted equation of daylight values have a whole number of minutes (the corrections are given in Section 4.4.14.1 of the Apparatus). We will see below that the remaining 17 differences of $\pm 30'''$ between extracted equation of daylight values and whole numbers of minutes are probably caused by deviations from the symmetry of the underlying right ascension table; the difference of $30''$ for argument 88 is probably due to an incorrect oblique ascension value. We can reconstruct the underlying table for the equation of daylight by rounding all extracted values to the nearest number of minutes.[68] The result is displayed in Table 4.2 (the reconstructed equation of daylight values are denoted by $T_\Delta(\lambda)$; the "error" column will be explained below). Note that, since the underlying equation of daylight has values to minutes only, the seconds' place of the oblique ascension values cannot generally be correct. This is reflected in the error statistics.

Since the tabular differences of the reconstructed equation of daylight table are constant over long stretches of the argument and are monotone decreasing, it seems probable that either linear interpolation or some other method of computation involving tabular differences was applied. Consequently, the tabular errors are probably correlated and the conditions for the least squares estimator as described in Section 2.4 may not be satisfied. If tabular values from the whole range of the argument are included, least squares estimation seems to point to the use of parameter values in the neighbourhood of $\varepsilon = 23;51$ and $\phi = 32;1$. As was indicated both in Section 2.4 and in Section 4.3.10, the confidence region for the obliquity of the ecliptic and the geographical latitude in a table for the equation of daylight has a highly oblong shape. Consequently, combinations of parameter

[67] Ordinarily we expect the unit of the extracted equation of daylight values in this case to be $30'''$.

[68] The best agreement with the pattern of first order tabular differences, with recomputed values for the plausible parameter values $\varepsilon = 23;35$ and $\phi = 32;20$ and with the reconstructed right ascension (see below) is obtained if the extracted value 16;1,30,0 for argument 88 is rounded to 16;2. We will see that the columns for the length of daylight and the hour length were based on the same values for the equation of daylight. Since these columns both contain a large number of errors, the reconstruction of the equation of daylight from the length of daylight or from the hour length would have been more problematic.

values lying outside the confidence region but close to the line through its long axis may lead to recomputations that are not much worse than the recomputation for the estimates found. In the present case, the plausible $\varepsilon = 23;35$, $\phi = 32;20$ is such a combination. If we apply a least squares estimation to only the first part of the reconstructed equation of daylight table, which is completely linear, we obtain nonsensical estimates for the obliquity ($\hat{\varepsilon} \approx 29$). If, on the other hand, we apply a least squares estimation to tabular values for arguments larger than 30, for which the tabular differences seem to be more reliable, we obtain smaller minimum obtainable standard deviations of the tabular errors and confidence regions containing the combination $\varepsilon = 23;35$, $\phi = 32;20$. I conclude that these were the parameter values for which the reconstructed equation of daylight was computed. The error statistics for the reconstructed table thus become $N = 90$, $n = 50$, $\mu = -15''$, $\sigma = 55'$. In particular close to argument 90 the agreement between the reconstructed equation of daylight and a recomputation is good (see the "error" column of Table 4.2). Below it will be shown that the same values for the equation of daylight were also used for the computation of the adjacent length of daylight and hour length columns.

As we now have reliable values for the equation of daylight underlying the oblique ascension in the first column of folios 143^{v}–149^{r}, we can easily reconstruct the underlying right ascension values from the formula $\alpha(\lambda) = \rho(\lambda) + \Delta(\lambda)$. Since we have seen that the oblique ascension table satisfies the symmetry $\rho(360-\lambda) = 360-\rho(\lambda)$, we need to consider the reconstructed right ascension only for arguments up to 180. We find 17 deviations of $\pm 1''$ from the symmetry $\alpha(180-\lambda) = 180-\alpha(\lambda)$, which correspond to the 17 extracted equation of daylight values that differ by $30'''$ from a whole number of minutes.

By means of a weighted estimation (see Section 2.2) or a least squares estimation it can be established beyond doubt that the reconstructed right ascension table is based on obliquity of the ecliptic $\varepsilon = 23;35$. In fact, if the outliers mentioned below are corrected, we find a 95 % confidence interval $\langle 23;34,59,14\,,\ 23;35,0,57 \rangle$ for the obliquity and a minimum obtainable standard deviation of $38'''$. The reconstructed right ascension has large errors for arguments 87 ($-1'$), 92 ($+1'$) and 93 ($+1'$). The errors for arguments 87 and 93 seem to be due to a symmetrical error in the original right ascension table ($T(87) = 86;42,37$ instead of the correct 86;43,37). The outlier for argument 88 seems to be related to the error of $30''$ in the extracted equation of daylight value for the same argument. Both errors disappear if we correct the oblique ascension value for argument 92 from 76;9,55 to 76;8,55. Note that this error (ح →ط) is not a common scribal mistake.

Apart from the outliers the reconstructed right ascension shows 22 errors of $+1''$ and 22 errors of $-1''$ for arguments between 1 and 90. Thus we conclude that it is less accurate than al-Baghdādī's right ascension table on folios 141^{v}–143^{r} and probably not related to it. I have not been able to find the reconstructed right ascension table in extant early zījes either.

4.3.14.2 Length of Daylight (2nd column)

Definition. The length of daylight $L(\lambda)$ is defined in Section 4.3.11.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
143v–149r, 2°	$1, 2, \ldots, 360$	0;0,8	23;35	32;20	90	72	$-2''$	$7''$

Analysis. It can easily be checked that the length of daylight column in the table on folios 143v–149r was computed from the inaccurate equation of daylight values used for the calculation of the oblique ascension in the adjacent column (these values are displayed in Table 4.2). The total number of differences between the length of daylight column and a recomputation from those values is only 23 (out of 360). Some of the differences are clearly scribal errors, others may be the result of careless calculation. In general the differences do *not* occur for symmetrical arguments, which implies that they cannot be the result of incorrect readings of equation of daylight values. All differences are listed in Section 4.4.14.2 of the Apparatus. For the error statistics I only corrected the obvious scribal error of $-40''$ in $T(59)$ = 13;46,16 (نو→يو).

4.3.14.3 Hour Length (3rd column)

Definition. The hour length $H(\lambda)$ is defined in Section 4.3.12.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
143v–149r, 3°	$1, 2, 3, \ldots, 360$	0;0,10	23;35	32;20	90	71	$-2''$	$8''$

Analysis. Like the length of daylight, the column for the hour length on folios 143v–149r can easily be seen to have been computed from the equation of daylight values underlying the oblique ascension column (see Table 4.2). If we recompute the hour length from those values, we obtain only 45 differences between the 360 hour length values in the manuscript and the recomputation. These differences are listed in Section 4.4.14.3 of the Apparatus. In the error statistics the hour length value for argument 59 was left out because it contains a large error, which is possibly the result of two consecutive scribal errors.

4.3.14.4 Solar Altitude (4th column)

Definition. See Section 4.3.6 for the definition of the solar altitude $h_s(\lambda)$.

folios	arguments	unit	ε	ϕ	N	n	μ	σ
143v–149r, 4°	$1, 2, \ldots, 360$	0;0,1	23;35	32;20	90	14	$-1''$	$6''$

Analysis. The solar altitude column in the table on folios 143v–149r of the Baghdādī Zīj was computed for obliquity of the ecliptic 23;35 and geographical latitude 32;20, as can be seen from $T(180)$ = 57;40,0 and $T(90)$ = 81;15,0. We can use the symmetry of the table to correct a number of scribal errors, which are listed in Section 4.4.14.4 of the Apparatus. By comparing the values in the manuscript with recomputed values, we find five more obvious scribal errors in the first quadrant of the table, also listed in the Apparatus. Since the same errors occur for the symmetrical arguments in the second quadrant, and since, in four of the five cases, the same errors with opposite sign occur also for the symmetrical arguments in the third and fourth quadrants, we can conclude that the cause of all these errors lies in mistakes in the underlying declination values. As three of the above-mentioned errors in the first quadrant (namely those for arguments $34, 40$ and 83) occur also in the declination table in the first column of folios 129^{v}–130^{v}, it seems probable that al-Baghdādī used that particular table for the calculation of the present solar altitude table. In fact, if we recompute the solar altitude by adding the declination values on folios 129^{v}–130^{v} to 57;40 and rounding to two sexagesimal fractional places in the modern way, we find only 22 differences in 360 tabular values.[69] The error statistics given above include the errors caused by scribal errors in the declination table. If these errors are corrected, the statistics become: $N = 90$, $n = 9$, $\mu = -3'''$, $\sigma = 22'''$.

4.4 Apparatus

For every table analysed in the previous section this Apparatus presents all corrections of scribal and other errors which could be made during the analysis. In most cases the corrections were made on the basis of mathematical properties of the tabulated function, e.g. the symmetry in the tabular values. Section 4.1.4 explains extensively how scribal and other errors can be corrected in this way. The analyses in the previous section provide more details about the method according to which the corrections in a particular table were made. Note that only incidentally were the corrections based on a comparison of al-Baghdādī's table with copies of the same table in other manuscripts.

For all tables, $T(x)$ will denote the tabular value for argument x as it occurs in the Baghdādī Zīj. The (scribal) error in a tabular value (tabular value minus corrected value) is displayed between round brackets: $(-40''')$ denotes an error of -40 sexagesimal thirds. The difference between a manuscript value and a recomputation (manuscript minus recomputation) is displayed between square brackets: $[+40''1''']$ denotes a difference of 40 seconds and 1 third. In the case of scribal errors the correction is also indicated in the Arabic *abjad* notation:[70] بو→نو means that the number بو occurring in the manuscript must be corrected to نو . Corrections that cannot be explained from a confusion of abjad numerals but could be confirmed in other ways are indicated by a "(?)".

[69] All these differences could be explained from confusion of the numerals 6 and 7 (و → ر) or from the use of truncation instead of modern rounding.

[70] See Section 1.1.1 for an explanation of the *abjad* notation.

4.4.1 Sine (folios 224ᵛ–225ᵛ)

The following scribal errors could be corrected by comparing the tabular values with recomputed values. The numbers between square brackets are the differences between manuscript and recomputation, the numbers between round brackets are the scribal errors.

$T(24)$	$=$ 24;24,15,17	[+10‴]	(+10‴)	ىر → ر
$T(26)$	$=$ 26;58, 8,10	[+40′]	(+40′)	نح → ح
$T(31)$	$=$ 30;54, 8,53	[+40‴]	(+40‴)	نج → ىج
$T(44)$	$=$ 41;40,46,33	[+21‴]	(+20‴)	لج → ىج
$T(49)$	$=$ 45;56,57,56	[+40′40‴]	(+40′40‴)	نو → ىو, نو → ىو
$T(75)$	$=$ 57;57, 9,59	[−10″]	(−10″)	ط → ىط

4.4.2 Versed Sine (folios 226ʳ–227ʳ)

The following scribal errors could be corrected in one of three ways: by using the symmetry of the table, by comparing the tabular values with the values in the preceding sine table or by comparing the tabular values with recomputed values. The numbers between square brackets are the differences between manuscript and recomputation. In this case these differences are identical to the actual scribal errors, given between round brackets.

$T(\ 18)$	$=$ 2;56,51,48	[+40″]	(+40″)	نا → ىا
$T(\ 48)$	$=$ 19;51, 7,40	[−7‴]	(−7‴)	م → مر
$T(\ 77)$	$=$ 46;30,10,32	[−2‴]	(−2‴)	لں → لد
$T(\ 96)$	$=$ 66;16,18,20	[+11‴]	(+11‴)	ک → ط
$T(118)$	$=$ 88;10, 5,11	[−40‴]	(−40‴)	ىا → نا
$T(119)$	$=$ 89; 5,18,13	[−40‴]	(−40‴)	ىج → نج
$T(121)$	$=$ 90;54, 8,53	[+40‴]	(+40‴)	نج → ىج
$T(147)$	$=$ 110;19,12,11	[−40‴]	(−40‴)	ىا → نا

4.4.3 Tangent and Cotangent

4.4.3.1 Cotangent (folios 227ᵛ–228ᵛ, 1ˢᵗ column)

The following scribal errors could be corrected by comparing the manuscript values with recomputed values or by comparing them with the values of the tangent in the adjacent column. In three cases (arguments $3, 82$ and 84) a comparison with the recomputation of the cotangent table from al-Baghdādī's sine table (see Section 4.3.3) was used to perform the correction. The numbers between square brackets are the differences between the manuscript and the recomputation from the sine table for arguments $3, 82$ and 84, the differences between manuscript and exact recomputation for all other arguments. The numbers between round brackets are the actual scribal errors.

$T(\ 3)$	$=$ 1144;52, 0,34	[−4″59‴]	(−5″)	ح → ه
$T(\ 4)$	$=$ 818; 2,23,33	[−40°22‴]	(−40°)	ضىح → ضنح
$T(\ 9)$	$=$ 370;49,30,6	[−8°14″]	(−8°)	شع → شعح
$T(24)$	$=$ 132;45,43,54	[−2°3‴]	(−2°)	قلں → قلد
$T(29)$	$=$ 108;14,34,57	[+38‴]	(+40‴)	نر → ىر
$T(65)$	$=$ 27;18,42,26,28	[−40′47ⁱᵛ]	(−40′)	ىح → نح

$T(73)$	=	18;20,34,49,13	$[-3''25^{iv}]$	$(-3'')$	لد → لر
$T(75)$	=	16; 4,34, 1,38	$[-2''59'''54^{iv}]$	$(-3'')$	لد → لر
$T(82)$	=	8;25,56,48,11	$[-40^{iv}]$	(-40^{iv})	ٮا → نا
$T(84)$	=	6;18,22,30,18	$[-41^{iv}]$	(-40^{iv})	ٮح → نح

4.4.3.2 Tangent (folios 227^{v}–228^{v}, 2nd column)

This table is essentially identical to the cotangent table in the adjacent column (see above). The scribal errors given below could be corrected by comparing the manuscript values with recomputed values or with the values of the cotangent table. The numbers between square brackets are the differences between manuscript and recomputation, the numbers between round brackets the actual scribal errors.

$T(17)$	=	18;20,34,49,13	$[-3''25^{iv}]$	$(-3'')$	لد → لر
$T(31)$	=	36; 8, 5,53,14	$[+4'59''59'''37^{iv}]$	$(+5')$	ح → حـ
$T(33)$	=	38;56,52, 2,12	$[-1'12^{iv}]$	$(-1')$	نو → نر
$T(34)$	=	40;28,53,51, 6	$[+40''43^{iv}]$	$(+40'')$	نحـ → ٮحـ
$T(38)$	=	46;12,37,42,26	$[-39'59''59'''17^{iv}]$	$(-40')$	ٮں → نں
$T(46)$	=	62; 7,14,32	$[-40''1''']$	$(+40'')$	ٮد → ند
$T(67)$	=	141;21, 4,52	$[+45''']$	$(+40''')$	نں → ٮں

4.4.3.3 Cotangent (folio 229^{r}, 1st column)

The following scribal errors were corrected on the basis of a comparison with the versions of this table found in four manuscripts of the Jāmiʿ Zīj, namely Istanbul Fatih 3418, folio 43^{v}; Cairo DM 188/2, folio 15^{r}; Leiden Or. 8 (1054), folio 35^{r}; and Berlin 5751, page 46. The corrections were confirmed by the recomputation of this table from a sine table with values calculated to three sexagesimal places (see Section 4.3.3).

$T(25)$	=	25;42	$(-2')$	مں → مد
$T(35)$	=	16; 8	(-1°)	ٮو → ٮر
$T(48)$	=	10;43	$(-5')$	محـ → مح
$T(57)$	=	7;43	$(-5')$	محـ → مح
$T(60)$	=	6;16	$(-40')$	ٮو → نو

4.4.3.4 Cotangent (folio 229^{r}, 2nd column)

The following scribal errors were corrected on the basis of a comparison with the versions of this table found in four manuscripts of the Jāmiʿ Zīj, namely Istanbul Fatih 3418, folio 43^{v}; Cairo DM 188/2, folio 15^{r}; Leiden Or. 8 (1054), folios 32^{r} and 35^{r}; and Berlin 5751, page 46. The corrections were confirmed by the recomputation of this table from the adjacent cotangent table for $R = 12$ (see Section 4.3.3).

$T(20)$	=	19;54	$(+40')$	ند → ٮد
$T(50)$	=	5;12	$(-40')$	ٮں → نں
$T(55)$	=	4;14	$(-40')$	ٮد → ند
$T(83)$	=	0;11	$(-40')$	ٮا → نا

4.4.4 Declination (folios 129v–130v, 1st column)

The following corrections of scribal errors are based on a comparison of the tabular values with recomputed values for obliquity $\varepsilon = 23;35$. The numbers between square brackets are the differences between manuscript and recomputation, the numbers between round brackets the actual scribal errors.

$T(9)$	= 3;35,57,53	[+40″ 1‴]	(+40″)	نر → ىر	
$T(34)$	= 12;55,48, 6	[+ 7″58‴]	(+8″)	مح → م	(?)
$T(40)$	= 14;54, 2,32	[− 4″59‴]	(−5″)	ٮ → ر	
$T(56)$	= 19;22,55,34	[+40″ 2‴]	(+40″)	نه → ٮه	
$T(57)$	= 19;36,58,49	[+40″ 2‴]	(+40″)	نح → ٮح	
$T(71)$	= 22;13,33,58	[− 5″ 6‴]	(−5″)	لح → لح	
$T(83)$	= 23;23,19,19	[−29″59‴]	(−30″)	ٮط → مط	

The errors in $T(29) = 11;11,2,56$ (−11‴), $T(72) = 22;21,53,13$ (+13‴) and $T(79) = 23;7,27,52$ (−39‴) are significantly larger than the other errors in the table, but cannot be identified as scribal errors and therefore were not corrected.

4.4.5 Second Declination (folios 129v–130v, 3rd column)

The following scribal errors could be corrected on the basis of a comparison with the versions of this table in Kushyār ibn Labbān's Jāmiʿ Zīj, available in the manuscripts Istanbul Fatih 3418, folios 83v–84r, 2nd col.; Cairo DM 188/2, folio 12r; Leiden Or. 8 (1054), folios 74v–75v, 2nd col.; and Berlin 5751, pages 144–146 4, 2nd col.

$T(42)$	= 16;16, 0	(−1′)	ٮو → ٮر
$T(58)$	= 20;18,18	(−40″)	ٮح → نح
$T(76)$	= 22;57,26	(−1″)	كو → كر
$T(77)$	= 23; 4,38	(+2′)	د → ٮ

4.4.6 Solar Altitude (folios 119v–120r)

The following corrections of scribal errors are based on a comparison of tabular values for symmetrical arguments and on a comparison with the declination table on folios 129v–130v, 1st column (see Section 4.3.4), from which the solar altitude values were computed.

$T(3)$	= 57;46,59,17	(+ 3‴)	ٮر → ٮد	
$T(19)$	= 64; 4, 3,21	(− 2‴)	كا → كح	(?)
$T(48)$	= 73;52,47,16	(−40‴)	ٮو → نو	
$T(51)$	= 74;41,53,25	(− 1‴)	كه → كو	
$T(55)$	= 75;42,11,31	(−40″)	ٮا → نا	
$T(72)$	= 78;56,53,53	(+40‴)	نح → ٮح	
$T(305)$	= 37;27,48,29	(+40‴)	مح → ح	
$T(316)$	= 40;26,49,32	(−20‴)	لٮ → نٮ	
$T(318)$	= 41; 3, 9,39	(−10″)	ط → ٮط	
$T(341)$	= 49; 5,26,37	(−30″)	كو → نو	
$T(346)$	= 51; 1,44, 6	(−30‴)	و → لو	
$T(359)$	= 56;10,59,55	(+10‴)	نه → مه	

4.4.7 Tangent of Declination (folio 235[r])

The following scribal errors could be corrected by comparing the manuscript values with recomputed values. The numbers between square brackets are the differences between manuscript and recomputation.

$T(17)$ =	7; 7, 0,44	[+2′59″58‴]	(+3′)	ر → د
$T(19)$ =	7;52,16,35	[−40″ 3‴]	(−40″)	ٮو → نو

The remaining outliers cannot be identified as scribal errors. They are likely to be the result of the particular method used for the computation of the table. The numbers between square brackets are the differences between manuscript and recomputation.

$T(\ 8)$ =	3;20,45,17	[−26‴]
$T(13)$ =	5;25,18,47	[−12‴]
$T(23)$ =	9;29,45,54	[−32‴]
$T(32)$ =	13; 1, 0, 4	[+21‴]
$T(39)$ =	15;36,35, 6	[+16‴]
$T(72)$ =	24;41,13,29	[+16‴]
$T(79)$ =	25;37,20,34	[−49‴]
$T(88)$ =	26;10,25,23	[+30‴]

4.4.8 Right Ascension (folios 141[v]–143[r])

The following scribal errors could be corrected on the basis of the symmetry of the table. The error for argument 328 can be explained from the fact that $T(329)$ also has the digit 48 in the thirds' place; apparently this digit was copied twice by mistake.

$T(\ 41)$ =	38;32,34,31	(− 3″)	لد → لر	
$T(\ 65)$ =	63; 1,18,10	(−40″)	ىح → نح	
$T(\ 68)$ =	66;12, 5,29	(−30″)	ه → له	
$T(\ 99)$ =	99;47,17,47	(− 1′)	مر → مح	(?)
$T(121)$ =	123;14,53,24	(− 5″)	نح → نح	
$T(144)$ =	146;20,32,32	(+ 2″)	لٮ → ل	
$T(148)$ =	150;12, 3,55	(+ 1″)	ح → ٮ	(?)
$T(178)$ =	178;10, 0,15	(−40‴)	ٮه → نه	
$T(193)$ =	191;57,48, 1	(+ 1′)	نر → نو	
$T(196)$ =	194;43,27, 5	(+ 1‴)	ه → د	
$T(265)$ =	264;32,49, 4	(+ 1‴)	د → ح	
$T(301)$ =	303;14,18,24	(−40″)	ىح → نح	
$T(302)$ =	304;16,12,48	(+ 1′)	ٮو → ىر	
$T(328)$ =	330;12, 2,48	(− 7‴)	مح → نه	(?)
$T(347)$ =	348; 3,51,59	(+40″)	نا → ىا	
$T(356)$ =	356;19,59,14	(− 1‴)	ٮد → ٮه	

The following two corrections of scribal errors are based on the recomputation of the oblique ascension table on folios 139^{v}–141^{r} from the right ascension table (for this recomputation see Section 4.3.13).

$T(27)$	= 25; 1,52, 4	$(-3''')$	د → ر
$T(207)$	= 205; 1,52, 4	$(-3''')$	د → ر

The only remaining deviations from the symmetry of the right ascension table are $T(4) = 3;40,0,46$ vs. $T(176) = 176;19,59,15$ and $T(184) = 183;40,0,46$ vs. $T(356) = 356;19,59,15$.

4.4.9 Sine of the Equation of Daylight (folio 235^{v})

The following scribal errors could be corrected by inspecting the first order differences of the tabular values (arguments 7, 9, 47 and 48) or by comparing the manuscript values with values recomputed by multiplying the tangent of declination values on folio 235^{r} by tan 33;25 (see Section 4.3.9). The numbers between brackets are the differences between manuscript and recomputation, the numbers between round brackets are the actual scribal errors.

$T(4)$	= 1; 6, 8,16	$[-40''']$	$(-40''')$	ٮو → نو
$T(7)$	= 1;55,17, 2	$[-40'']$	$(-40'')$	ٮر → نر
$T(9)$	= 2;28,17, 7	$[-40'']$	$(-40'')$	ٮر → نر
$T(25)$	= 6;47,28,13	$[-40''']$	$(-40''')$	ىح → نح
$T(29)$	= 7;49,38, 7	$[+3''']$	$(+3''')$	ر → د
$T(33)$	= 8;50,18,12	$[-40''']$	$(-40''')$	ٮٮ → نٮ
$T(47)$	= 11; 6,48,54	$[-1°]$	$(-1°)$	ٮا → ٮٮ
$T(48)$	= 11;59,39,37	$[-20']$	$(-20')$	ٮا → ٮٮ (?), نط → ٮط
$T(49)$	= 12;32,18,19	$[-40''']$	$(-40''')$	ٮط → نط
$T(57)$	= 14; 6, 2,58	$[+40''']$	$(+40''')$	نح → ىح
$T(73)$	= 16;23,36,18	$[-40''']$	$(-40''')$	ىح → نح
$T(74)$	= 16;29,35,18	$[-40''']$	$(-40''')$	ىح → نح
$T(86)$	= 17;13,53,13	$[-41''']$	$(-40''')$	ىح → نح

4.4.10 Equation of Daylight (folio 117^{r})

The following scribal errors could be corrected by inspecting the first order differences of the tabular values.

$T(15)$	= 3;56,59,36	$(+40'')$	نط → ٮط
$T(16)$	= 4;11,13,32	$(-40'')$	ىح → نح
$T(27)$	= 6;59,15,45	$(-44'')$	ٮه → نط (?)
$T(30)$	= 7;45,30,29	$(+1')$	مه → مد
$T(47)$	= 11;38,11,45	$(-40'')$	ٮا → نا
$T(71)$	= 15;33,31,41	$(-5')$	لح → لح
$T(81)$	= 16;29,38,21	$(+20'')$	لح → ىح

The following corrections of scribal errors are based on the recomputation of the oblique ascension table on folios 139ᵛ–141ʳ from the table for the equation of daylight (see Section 4.3.13):

$T(16)$	=	4;11,13,32	$(-5''')$	لب → لر
$T(37)$	=	9;25,13,47	$(+7''')$	مر → م
$T(67)$	=	15; 9, 2,27	$(+3''')$	كر → كد
$T(79)$	=	16;21,54,15	$(-40''')$	به → نه

4.4.11 Length of Daylight (folios 117ᵛ–118ʳ)

The following scribal errors could be corrected on the basis of the symmetry of the table and by comparing the tabular values with the copy of this table in the zīj of Ḥabash al-Ḥāsib (Berlin 5750, folios 112ʳ–112ᵛ). The corrections were confirmed by the recomputation of the length of daylight table from the table for the equation of daylight (see Section 4.3.11).

$T(\ 35)$	=	13;11,34,39	$(-1'')$	لد → له
$T(\ 45)$	=	13;29,46,26	$(-1''')$	كو → كر
$T(\ 50)$	=	13;38, 2,20	$(-5'')$	ب → ر
$T(\ 58)$	=	13;50, 8,57	$(+40''')$	نر → ٮر
$T(\ 67)$	=	14; 1,12,59	$(+40''')$	نط → ٮط
$T(\ 72)$	=	14; 6, 5,57	$(+5'')$	ه → o̅
$T(\ 79)$	=	14;10,55,59	$(+40''')$	نط → ٮط
$T(276)$	=	9;46,58,33	$(-5''')$	لج → لح
$T(278)$	=	9;47,40,56	$(+40''')$	نو → ٮو
$T(342)$	=	11;22,16,48	$(+5''')$	مح → مج

After the correction of the scribal errors we find the following differences between the tabular values and the recomputation from the equation of daylight. Since the corrected length of daylight table is symmetrical except for arguments 9 and 351, only the differences for arguments between 1 and 90 and for argument 351 are given. Note that the symmetrical differences for arguments between 271 and 360 have opposite signs. The difference for argument 65 is much larger than all other differences and may be due to a computational mistake.

$T(\ \ 1)$	=	12; 2, 6,43	$[+1''']$
$T(\ \ 9)$	=	12;18,58,30	$[+16''']$
$T(\ 52)$	=	13;41,17,30	$[-1''']$
$T(\ 53)$	=	13;42,50,17	$[-1''']$
$T(\ 56)$	=	13;47,18,29	$[+2''']$
$T(\ 65)$	=	13;58,59, 0	$[-1'']$
$T(\ 82)$	=	14;12,19,44	$[+3''']$
$T(\ 85)$	=	14;13,17,49	$[+2''']$
$T(351)$	=	11;41, 1,38	$[-8''']$

4.4.12 Hour Length (folios 118^{v}–119^{r})

The following scribal errors could be corrected on the basis of the symmetry of the table and by comparing the tabular values with the copy of this table in the Zīj of Ḥabash al-Ḥāsib (Berlin 5750, folios 109^{r}–109^{v}). The corrections were confirmed by the recomputation of this table from the table for the equation of daylight (see Section 4.3.12).

$T(2) = 15;\ 5,16,42$	$(-2''')$	مب → مد	
$T(24) = 16;\ 2,28,13$	$(-5''')$	ىح → ىح	
$T(30) = 16;16,25,\ 5$	$(-1')$	ىو → ىر	
$T(32) = 16;23,57,43$	$(+1'39''59''')$	كح → كب , نر → ىر, ىح → مد	(?)
$T(33) = 16;24,42,\ 8$	$(-30''')$	ح → لح	
$T(85) = 17;46,47,16$	$(+10'')$	مر → لر	
$T(292) = 12;27,11,11$	$(-1''')$	ىا → س	(?)
$T(312) = 13;\ 1,24,\ 5$	$(-2'')$	كد → كو	
$T(313) = 13;\ 3,31,21$	$(-1''')$	كا → كب	(?)
$T(326) = 13;32,13,24$	$(-40'')$	ىح → نح	
$T(334) = 13;52,29,15$	$(-40''')$	ىه → نه	
$T(350) = 14;33,39,42$	$(-7''')$	مب → مط	(?)
$T(357) = 14;52,\ 4,17$	$(-40''')$	ىر → نر	

After the correction of the scribal errors we find the following differences between the tabular values and the recomputation from the table for the equation of daylight. Since the corrected table is symmetrical except for arguments 9 and 351, only the differences for arguments between 1 and 90 and for argument 351 are given. Note that the symmetrical differences for arguments between 351 and 360 have opposite signs.

$T(9) = 15;23,43,\ 7$	$[+19''']$
$T(52) = 17;\ 6,36,53$	$[-1''']$
$T(53) = 17;\ 8,32,51$	$[-2''']$
$T(56) = 17;14,\ 8,\ 7$	$[+3''']$
$T(82) = 17;45,24,41$	$[+5''']$
$T(85) = 17;46,37,16$	$[+2''']$
$T(351) = 14;36,17,\ 3$	$[-9''']$

4.4.13 Oblique Ascension

4.4.13.1 Latitude of Baghdad (folios 139^{v}–141^{r})

The following scribal errors could be corrected by comparing the extracted right ascension and equation of daylight tables with the tables for these functions elsewhere in the Baghdādī Zīj. The corrections for arguments 32, 44, 47, 104, 107, 110, 132, 133, 136 and 152 are confirmed by deviations from the symmetry of the oblique ascension. For the other arguments the symmetrical tabular value is in error as well. The errors for arguments 47 and 133 could have resulted from a misreading of the correct right ascension value $T(47) = 44;30,11,2$ as $44;30,51,2$ (ىا→ىا). See Sections 4.4.8 and 4.4.10 for corrections of al-Baghdādī's tables for the right ascension and the equation of daylight based on the comparison with the tables extracted from the oblique ascension table.

$T(4)$ =	2;36,40,17	$(-40''')$	ىر → نر
$T(32)$ =	21;34,11,42	$(+1'')$	ىا → ے
$T(44)$ =	30;30,58,30	$(+40'')$	نح → ىح
$T(47)$ =	32;51,59,17	$(+40'')$	نط → لط
$T(77)$ =	59;38,32,29	$(-20''')$	كط → مط
$T(96)$ =	79;54,51,16	$(-30''')$	ىو → مو
$T(104)$ =	89; 4,16,42	$(-40'')$	ىو → نو
$T(107)$ =	92;35, 7, 0	$(-24'')$	ر → لا (?)
$T(110)$ =	96; 7,59,33	$(+20'')$	لح → ىح
$T(120)$ =	108; 6,16, 2	$(+18''')$	ىو → ىه, ٮ → مد
$T(132)$ =	122;38,11,29	$(-30''')$	كط → نط
$T(133)$ =	123;50,17,53	$(-39''20''')$	ىر → نر, نح → ىح
$T(136)$ =	127;29, 6,16	$(-40''')$	ىو → نو
$T(152)$ =	146;46,16,16	$(-1'')$	ىو → ىر
$T(157)$ =	152;44,57,10	$(+10''')$	ے → ع (?)
$T(163)$ =	159;53,25, 5	$(+3''')$	ه → ٮ (?)

The following scribal errors in the second half of the oblique ascension table could be corrected on the basis of the symmetry of the table.

$T(191)$ =	192;59,47,58	$(+5''')$	نح → ىح
$T(205)$ =	209;38,20, 7	$(-1'')$	ڪ → كا (?)
$T(238)$ =	249;28,59,16	$(+10'')$	نط → مط
$T(247)$ =	260;17,57,11	$(+1'')$	نر → نو
$T(271)$ =	287;49,40,47	$(-10''')$	مر → نر
$T(286)$ =	303;19,44, 8	$(-1'')$	مد → مه
$T(295)$ =	311;50,31, 8	$(-40''')$	ح → مح
$T(309)$ =	323;55,42,12	$(-10''')$	س → كٮ
$T(344)$ =	349;28,27,33	$(+1'')$	كر → كو

4.4.13.2 Latitudes 30° to 40° (folios 150ᵛ–160ʳ)

Latitude 30° **(folios 150ᵛ–151ʳ).** The following deviation from the symmetry of the oblique ascension could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. The corrections of four other deviations would eliminate one error in the two extracted tables and four of the eight errors in the oblique ascension table itself, but could not be confirmed in any other way.

$$T(30) = 21;\ 2 \qquad (-5') \qquad ٮ \to ر$$

Latitude 31° **(folios 151ᵛ–152ʳ).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Four other corrections of deviations would eliminate a number of errors in the oblique ascension table, but could not be confirmed in any other way.

$T(\ \ 4)$	$= 2;44$	$(+2')$	مد → مٮ
$T(\ 97)$	$= 82;37$	$(+3')$	لر → لد
$T(251)$	$= 263;30$	$(-7')$	ل → لر
$T(334)$	$= 342;\ 7$	$(+3')$	ر → د
$T(356)$	$= 357;58$	$(+40')$	نح → ىح

Latitude 33° **(folios 152ᵛ–153ʳ).** This table includes 13 deviations of $\pm 1'$ from the symmetry of the oblique ascension. The correction of some of these deviations would eliminate a couple of errors in the oblique ascension table, but could not be confirmed in any other way.

Latitude 34° **(folios 153ᵛ–154ʳ).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Four other corrections of deviations would eliminate a number of errors in the oblique ascension table, but could not be confirmed in any other way.

$T(301)$	$= 317;32$	$(+2')$	لٮ → ل
$T(330)$	$= 340;\ 7$	$(+5')$	ر → ٮ

Latitude 35° **(folios 154ᵛ–155ʳ).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Three other corrections of deviations would eliminate some of the errors in the oblique ascension table, but could not be confirmed in any other way.

$T(\ 30)$	$= 19;38$	$(-2')$	لح → م	(?)
$T(104)$	$= 88;\ 2$	$(-2')$	ٮ → د	

Latitude 36° (folios 155ᵛ–156ʳ). The following deviations from the symmetry of the oblique ascension could all be corrected after we had noted that the underlying right ascension and equation of daylight had values to minutes. The corrections are such that the digit in the second's place of all extracted values becomes 0. Only in the case of argument 294 was it necessary to check the first order differences of the extracted tables. All corrections were confirmed by an inspection of the copies of this table in Kushyār ibn Labbān's Jāmiʿ Zīj.

$T(\ 87)$	$= 68;17$	$(-1')$	ىر → ىح	(?)
$T(\ 92)$	$= 73;43$	$(-1')$	مح → مد	
$T(103)$	$= 86;14$	$(-1')$	ىد → ىه	
$T(104)$	$= 87;25$	$(-1')$	كه → كو	
$T(106)$	$= 89;47$	$(+1')$	مر → مو	
$T(148)$	$= 141;\ 8$	$(+1')$	ح → ر	(?)
$T(167)$	$= 164;16$	$(-1')$	ىو → ىر	
$T(196)$	$= 199;21$	$(-1')$	كا → كٮ	(?)
$T(200)$	$= 204;13$	$(-1')$	ىح → ىد	
$T(281)$	$= 300;\ 3$	$(-1')$	ح → د	
$T(290)$	$= 308;48$	$(+1')$	مح → مر	(?)
$T(294)$	$= 312;32$	$(+2')$	لٮ → ل	

Latitude 37° **(folios 156^{v}–157^{r}).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Seven other corrections of deviations would eliminate a large number of errors in the oblique ascension table, but could not be confirmed in any other way.

$T(216) = 224;12$	$(+2')$	ىب → ى	
$T(217) = 225;26$	$(+2')$	كو → كد	

Latitude 38° **(folios 157^{v}–158^{r}).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Seven other deviations of $\pm 1'$ could not be corrected.

$T(\ 28) = 17;27$	$(+3')$	كز → كد	
$T(\ 96) = 76;49$	$(+4')$	مط → مه	
$T(147) = 139;\ 9$	$(-4')$	ط → يج	(?)
$T(332) = 342;37$	$(+1')$	لز → لو	

Latitude 39° **(folios 158^{v}–159^{r}).** The following deviations from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Two other deviations of $\pm 1'$ could not be corrected.

$T(\ 56) = 37;\ 2$	$(-5')$	ب → ز	
$T(146) = 137;37$	$(+3')$	لز → لد	

Latitude 40° **(folios 159^{v}–160^{r}).** The following deviation from the symmetry could be corrected by inspecting the tabular differences or by comparing the table with recomputed values. Ten other deviations of $\pm 1'$ could not be corrected.

$T(307) = 325;11$	$(-40')$	ىا → نا	

4.4.14 The table attributed to al-Baghdādī on folios 143^{v}–149^{r}

4.4.14.1 Oblique ascension (1st column)

The following deviations from the symmetry of the oblique ascension could be corrected on the basis of our conclusion that the underlying equation of daylight table has values to minutes only (see Section 4.3.14.1).

$T(\ \ 5) = 3;20,\ 3$	$(-1'')$	ج → د	
$T(\ 16) = 10;43,23$	$(-5'')$	كج → كح	
$T(\ 32) = 21;54,57$	$(+1'')$	نز → نو	
$T(141) = 133;58,\ 8$	$(+1')$	نح → نز	(?)
$T(162) = 158;55,\ 4$	$(+1'')$	د → ج	
$T(210) = 215;18,\ 4$	$(-1'')$	د → ه	
$T(212) = 217;40,57$	$(+1'')$	نز → نو	
$T(324) = 335;\ 9,34$	$(+3'')$	لد → لا	

4.4.14.2 Length of Daylight (2nd column)

We find the following differences between the values of this table and length of daylight values recomputed from the equation of daylight given in Table 4.2. I assume that most of the differences are a result of careless calculation.

error $-1'$	for arguments	137
$-40''$		$59, 121, 258$
$-32''$		145
$-8''$		$65, 104, 206$
$-4''$		$30, 92, 141$
$+4''$		$87, 261, 268, 272, 321$
$+8''$		$56, 109$
$+12''$		208
$+20''$		311
$+40''$		139
$+1''$		$116, 244$

4.4.14.3 Hour Length (3nd column)

We find the following differences between the values of this table and hour length values recomputed from the equation of daylight given in Table 4.2. Again most of the differences are probably a result of careless calculation. Note that the errors of $-5''$ and $+5''$ correspond precisely to the errors of $-4''$ and $+4''$ in the length of daylight column (see above).

error $-20''$	for arguments	$223, 233, 242, 307, 317$
$-10''$		$65, 126, 127, 128, 206, 306$
$-5''$		$30, 92, 141$
$+5''$		$87, 261, 268, 272, 321$
$+10''$		$43, 53, 56, 99, 102, 109, 118, 137, 138, 144, 147,$
		$150, 209, 212, 215, 255, 257, 259, 292, 311, 316$
$+20''$		239
$+50''$		139
$+1''$		327
$+1'10''$		59
$+10'$		179

4.4.14.4 Solar Altitude (4th column)

The following scribal errors could be corrected on the basis of the symmetry of the table:

$T(100) = 80;52,13$	$(-\ 1'')$	يج → يد	
$T(163) = 64;23,\ 2$	$(-\ 1'')$	ب → ح	(?)
$T(268) = 34;\ 5,15$	$(-40'')$	يه → نه	
$T(284) = 34;49,26$	$(-\ 1'')$	كو → كر	
$T(314) = 40;56,27$	$(+\ 1'')$	كر → كو	

The following errors were corrected on the basis of a comparison with recomputed values. The same errors occur for the symmetrical arguments in the second quadrant. Futhermore, apart from the error for argument 40, they also occur with opposite sign for the symmetrical arguments in the third and fourth quadrants. From this we can conclude that all these errors are likely to derive from errors in the underlying solar declination table. The numbers between square brackets are the differences between manuscript and recomputation, the numbers between round brackets the actual scribal errors.

$T(\ 5) = 59;34,14$	$[-40'']$	$(-40'')$	ٮد → ند	
$T(34) = 70;35,48$	$[+\ 8'']$	$(+8'')$	مح → م	(?)
$T(40) = 72;34,\ 3$	$[-\ 5'']$	$(-5'')$	حـ → ح	
$T(47) = 74;40,20$	$[-30'']$	$(-30'')$	ك → ن	
$T(83) = 81;\ 3,19$	$[-30'']$	$(-30'')$	ٮط → مط	

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Index

Subjects

Names

Titles

Parameter Values

Manuscripts consulted

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Samenvatting

De Griekse astronoom Claudius Ptolemeus (ca. 150 na Christus) ontwikkelde meetkundige modellen waarmee de schijnbare bewegingen van de zon, de maan en de vijf toendertijd bekende planeten nauwkeurig konden worden voorspeld. Hij berekende een groot aantal tabellen voor niet-triviale wiskundige functies waarmee de posities van de hemellichamen via enkele optellingen en vermenigvuldigingen konden worden berekend. In de *Almagest* beschreef Ptolemeus zijn modellen en legde hij de berekening en het gebruik van zijn tabellen uit.

Na de snelle expansie van de Islam in de zevende en achtste eeuw begonnen de kaliefen in Baghdad de ontwikkeling van de exacte wetenschappen te bevorderen. Aanvankelijk werden Indiase en Perzische bronnen bestudeerd, maar al spoedig werden op grote schaal Griekse werken van klassieke auteurs in het Arabisch vertaald. Daarbij werden ook de werken van Ptolemeus herontdekt. Islamitische astronomen kwamen tot de conclusie dat posities van de zon, maan en planeten berekend met behulp van de tabellen in de *Almagest* niet meer nauwkeurig waren. Hoewel Ptolemeus' modellen nog goed voldeden, bleken de waarden van sommige astronomische parameters (bijvoorbeeld de helling van de ecliptica en de eccentriciteit van de zon) in de loop der eeuwen te zijn veranderd, terwijl de nauwkeurigheid van andere parameterwaarden (met name de middelbare bewegingen van de planeten) onvoldoende was om na 600 jaar nog steeds juiste posities te geven.

Islamitische astronomen begonnen daarom eigen waarnemingen van de bewegingen van de hemellichamen te doen en bepaalden nieuwe waarden van de parameters van de planeetmodellen. Op basis van de resultaten stelden ze astronomische handboeken samen, die in het Arabisch *zīj* werden genoemd en bestonden uit verzamelingen tabellen en verklarende tekst. In totaal werden tussen de achtste en de vijftiende eeuw door islamitische astronomen meer dan 200 verschillende zījes geschreven. In veel van die zījes werden nieuwe waarden voor de onderliggende parameters en nauwkeuriger berekeningsmethodes voor de tabellen gebruikt.

Astronomie speelde een belangrijke rol in de islamitische exacte wetenschappen in het algemeen en in de wiskunde in het bijzonder. Geavanceerde meetkunde, boldriehoeksmeting en numerieke methoden werden vooral toegepast in planeetmodellen en in de berekening van astronomische tabellen. Daarom vormen zījes één van de belangrijkste thema's van het onderzoek in de geschiedenis van de islamitische exacte wetenschappen. Ten einde de ontwikkeling van de islamitische astronomie vast te leggen zijn we met name geïnteresseerd in de antwoorden op de volgende vragen:

- Welke astronomen deden eigen waarnemingen?

- Welke astronomen voerden nieuwe waarden voor astronomische parameters en nieuwe berekeningsmethoden in?
- Welke verbanden bestaan er tussen de talloze islamitische zījes?

In totaal zijn duizenden handschriften van zījes bewaard gebleven, die zijn verspreid over bibliotheken in de hele wereld, waaronder Nederland. Slechts enkele tientallen van deze handschriften zijn tot nu toe systematisch onderzocht. In handschriften van zījes vinden we vaak materiaal (zowel tabellen als verklarende tekst) afkomstig van verschillende auteurs. Sommige zījes werden geschreven door materiaal uit verschillende vroegere bronnen samen te voegen. Andere zījes, die oorspronkelijk materiaal van één bepaalde astronoom bevatten, zijn uitsluitend bewaard gebleven in bewerkingen met talloze toevoegingen uit latere bronnen. In de meeste gevallen zijn de handschriften niet gedateerd en vermeldt de verklarende tekst niet van welke auteurs de tabellen afkomstig zijn.

We kunnen aannemen dat de tabellen van één en dezelfde astronoom bepaalde wiskundige eigenschappen gemeenschappelijk hebben, bijvoorbeeld de parameterwaarden, de berekeningsmethoden en gebruikte hulptabellen. Daarom kunnen de wiskundige eigenschappen van een tabel informatie geven over de herkomst van die tabel. Echter, ook de wiskundige eigenschappen van tabellen in zījes worden meestal niet of onjuist in de verklarende tekst vermeld. Herberekeningen volgens de aangegeven berekeningsmethoden komen meestal niet overeen met de tabellen in de zījes en de vermelde parameterwaarden zijn niet altijd daadwerkelijk voor de berekeningen gebruikt. We concluderen daarom dat het zeer nuttig is methoden tot onze beschikking te hebben waarmee de wiskundige eigenschappen van een tabel uit de tabelwaarden alleen kunnen worden gedestilleerd. Tot voor kort is daarbij nog geen intensief gebruik gemaakt van geavanceerde wiskundige en statistische methoden en computers.

In dit proefschrift worden diverse statistische methoden beschreven waarmee de parameterwaarden en de gebruikte berekeningswijze van astronomische tabellen kunnen worden vastgesteld. Met name worden in Hoofdstuk 2 de volgende vier schatters voor de waarden van de parameters in een gegeven tabel beschreven:

1. **Gewogen schatter.** In veel gevallen waarin slechts één van de parameterwaarden van een tabel onbekend is, kan deze waarde rechtstreeks worden geschat uit één enkele tabelwaarde. De nauwkeurigheid van de zo verkregen benadering is in sommige delen van de tabel aanzienlijk groter dan in andere delen. Bovendien kunnen grote schrijf- of rekenfouten in de gebruikte tabelwaarde de benadering onbruikbaar maken. Een aanzienlijk betere benadering van de onbekende parameterwaarde kan berekend worden door de benaderingen uit alle afzonderlijke tabelwaarden op een dusdanige manier te middelen dat de nauwkeurigste afzonderlijke schattingen het zwaarst meewegen. Het aldus verkregen gewogen gemiddelde noem ik de "gewogen schatter" voor een onbekende parameterwaarde. In § 2.2.1 worden de onzuiverheid en de variantie van de gewogen schatter berekend, alsmede een 95 % betrouwbaarheidsgebied voor de onbekende parameter. In § 2.A.2 wordt met behulp van de centrale limietstelling aangetoond dat de gewogen schatter bij benadering een normale verdeling heeft.
2. **Fourierschatter.** Met behulp van de Fourierschatter kan een translatieparameter geschat worden uit een tabel voor een funktie die aan bepaalde symmetrierelaties

voldoet. Een voorbeeld van zo'n functie is de middelpuntsvereffening, waarvoor het apogeum van de zon een translatieparameter is. De Fourierschatter is gebaseerd op benaderingen van de Fouriercoëfficiënten van de functie, die uit de tabelwaarden worden berekend. Hiervoor is het niet nodig de getabelleerde functie precies te kennen. In § 2.3 worden de onzuiverheid en variantie van de Fourierschatter bepaald. De verdeling van de Fourierschatter blijkt in de praktijk bij benadering normaal te zijn. In enkele speciale gevallen verloopt de berekening van de nauwkeurigheid van de Fourierschatter iets anders; zo blijkt de schatter ontaard te zijn als de waarde van de translatieparameter samenvalt met een van de argumenten van de tabel of precies tussen twee argumenten in valt.

3. **Kleinste-kwadratenschatter.** Met name voor tabellen met meerdere onbekende parameterwaarden bewijst de methode van kleinste kwadraten goede diensten. Het blijkt dat de onbekende parameters met behulp van de Gauss-Newton optimaliseringsmethode in het algemeen in een klein aantal stappen goed kunnen worden benaderd. Uit de benaderde covariantiematrix van de schattingen kunnen een betrouwbaarheidsgebied en afzonderlijke betrouwbaarheidsintervallen voor de onbekende parameters worden berekend.

4. **Kleinste-aantal-fouten-criterium.** Volgens dit criterium worden de onbekende parameterwaarden zodanig bepaald dat het aantal fouten in de tabel geminimaliseerd wordt. Voor het geval van een enkele onbekende parameter levert dit een interval van mogelijke parameterwaarden op. § 2.5 beschrijft de manier waarop aan het kleinste-aantal-fouten-criterium een statistisch fundament kan worden gegeven; de gevonden benaderingen van de onbekende parameter zijn dan meest aannemelijke (maximum likelihood) schattingen.

In de talloze voorbeelden van analyses van astronomische tabellen in dit proefschrift worden ook veel "ad hoc methodes" gebruikt die berusten op bepaalde wiskundige eigenschappen van een tabel zoals symmetrie of continuïteit van de getabelleerde functie. In § 4.1.4 wordt een aantal van zulke methodes beschreven, die vooral gebruikt kunnen worden voor het corrigeren van schrijffouten. In diverse voorbeelden in de Hoofdstukken 2 tot en met 4 wordt aangetoond dat de gecombineerde toepassing van de vier schatters voor onbekende parameterwaarden en ad hoc methodes het mogelijk maakt zowel de onbekende parameterwaarden als de gebruikte berekeningsmethode van tabellen voor gecompliceerde functies te bepalen. In meerdere gevallen kunnen uit de gevonden wiskundige eigenschappen conclusies omtrent de herkomst van de tabellen worden getrokken.

Met name worden in dit proefschrift de volgende voorbeelden van analyses van astronomische tabellen behandeld:

- Ter illustratie van het gebruik van de hierboven uitgelegde parameterschatters worden in § 2.6 drie tabellen uit drie verschillende zījes geanalyseerd. In één van de tabellen voor de schuine klimming in de Sanjufīnī Zīj, samengesteld door een Arabische astronoom in Tibet in de 14^{e} eeuw, vinden we twee verschillende waarden voor de helling van de ecliptica. Van de middelpuntsvereffeningtabel in de Shāmil Zīj (Noord-Perzië, 13^{e} eeuw) bepalen we de onderliggende eccentriciteit van de zon met grote nauwkeurigheid. Van een tabel in de Jāmiʿ Zīj van Kushyār ibn Labbān (Baghdad, ca. 1000) vinden we dankzij het gebruik van benaderde Fouriercoëfficiënten en van de kleinste-kwadratenschatter de precieze wiskundige structuur. Het blijkt dat de tabel, die de ware positie van de zon geeft, gebaseerd is op een benaderingsmethode voor de middelpuntsvereffening en op parameterwaarden die elders worden toegeschreven aan Yaḥyā ibn Abī Manṣūr (Baghdad, ca. 830), één van de belangrijkste vroeg-islamitische astronomen.

- In Hoofdstuk 3 worden vier tabellen voor de tijdsvereffening geanalyseerd. De tijdsvereffening is een ingewikkelde functie die van vier verschillende parameters afhangt en op verschillende manieren werd getabelleerd. Behalve in situaties waarin de parameterwaarden reeds bekend zijn is het practisch onmogelijk zonder geavanceerde wiskunde de berekeningswijze van een tijdsvereffeningtabel te bepalen. In dit proefschrift worden na een technische uitleg de tijdsvereffeningtabellen van Ptolemeus, Kushyār ibn Labbān (zie boven) en al-Baghdādī (zie onder) geanalyseerd. Uit de analyses blijkt dat Ptolemeus op meerdere manieren de berekening van zijn tijdsvereffeningtabel in de *Handige Tabellen* vereenvoudigde. Zo paste hij ten eerste lineaire interpolatie toe om de benodigde rechte-klimmingwaarden te berekenen. Vervolgens rondde hij zijn waarde voor het apogeum van de zon, die in alle bronnen als $65°30'$ wordt gegeven, af op $66°$ om de berekening van de benodigde waarden voor de middelpuntsvereffening te vereenvoudigen. Tenslotte gebruikte hij lineaire interpolatie in intervallen van zes graden om niet alle tijdsvereffeningwaarden exact te hoeven berekenen. Op deze manier bereikte Ptolemeus een enorme besparing aan rekenwerk zonder dat zijn tabel belangrijk aan nauwkeurigheid verloor.
- In Hoofdstuk 4 worden vrijwel alle goniometrische en sferisch-astronomische tabellen uit de zīj van al-Baghdādī (Baghdad, ca. 1285) geanalyseerd. Van de meeste tabellen wordt de gevolgde berekeningsmethode vastgesteld. Het blijkt dat al-Baghdādī materiaal kopiëerde van diverse vroegere astronomen, met name van Ḥabash al-Ḥāsib (Baghdad, ca. 850) en van Kushyār ibn Labbān (zie boven). Van een verzameling zeer nauwkeurige tabellen die een duidelijke samenhang vertonen, vinden we aanwijzingen dat ze afkomstig zijn van Abu'l-Wafāʾ (Baghdad, ca. 970), wiens tabellen niet in andere handschriften bewaard zijn gebleven.

Om het onderzoek in dit proefschrift uit te kunnen voeren heb ik diverse omvangrijke programma's voor de Personal Computer geschreven. Met behulp van het programma TABLE-ANALYSIS, beschreven in § 1.4.1, kunnen tabellen worden ingelezen, opgeslagen, bewerkt, afgedrukt en wiskundig geanalyseerd. Met drie andere programma's kunnen tabellen voor de middelbare bewegingen van planeten worden geanalyseerd, data in verschillende kalenders worden omgerekend en berekeningen met getallen in het zestigtallig stelsel worden uitgevoerd.

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The Bibliothèque Nationale in Paris kindly gave me permission to include photographs of folio 38^{v} of their manuscript Arabe 6040 (the zīj of al-Sanjufīnī), of folio 26^{r} of the manuscript Arabe 2528 (the Shāmil Zīj) and of folios 143^{v}–144^{r} of the manuscript Arabe 2486 (the Baghdādī Zīj). The Universiteitsbibliotheek of Leiden kindly agreed to let me include a photograph of folio 76^{v} of their manuscript BPG 78 (Ptolemy's *Handy Tables*).

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Curriculum Vitae

De auteur van dit proefschrift werd op 4 maart 1962 geboren te Haarlem. Van 1974 tot 1980 volgde hij het O.V.W.O. aan het Goois Lyceum te Bussum. In september 1980 begon hij aan de Universiteit Utrecht met de studie wiskunde. Op 30 mei 1983 behaalde hij het kandidaatsdiploma (cum laude) en op 30 november 1987 het doctoraaldiploma. De auteur volgde een groot bijvak informatica en maakte tijdens zijn afstudeerfase bij Prof. H.J.M. Bos (geschiedenis van de wiskunde) een begin met het onderzoek beschreven in dit proefschrift. Hij vervulde tijdens zijn studie meerdere studentassistenschappen.

Van 1 december 1987 tot 1 januari 1989 was de auteur als assistent in opleiding in dienst bij het Mathematisch Instituut van de Universiteit Utrecht. Een beurs van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (N.W.O.) stelde hem in staat vervolgens gedurende tien maanden onderzoek te doen aan het Institut für Geschichte der Naturwissenschaften te Frankfurt am Main. Van 1 december 1989 tot 1 juli 1993 was de auteur in het kader van een project van N.W.O. (aanstelling als wetenschappelijk medewerker) opnieuw werkzaam aan het Mathematisch Instituut te Utrecht. In deze periode gaf hij wiskunde-onderwijs en deed hij het grootste deel van het in dit proefschrift neergelegde onderzoek.