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A table for the true solar longitude in the Jāmi[c] Zīj

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In this article I analyse a table that turns out to be related to a table investigated by Prof. E.S. Kennedy in the Festschrift for Willy Hartner (1977). The main part of this article is a reworked version of Section 2.6.3 of my doctoral thesis (van Dalen 1993 in the Bibliography).

Between the eighth and fifteenth centuries Islamic astronomers compiled more than 200 different astronomical handbooks, known by the name of *zīj*. Most of these zījes contained explanatory text and large sets of tables of complicated mathematical functions, by means of which the positions of the sun, moon and planets could be accurately predicted.[1] The Islamic astronomers mostly based their zījes on Ptolemy's planetary models, but they calculated the tables anew, using more accurate methods of computation and more accurate values of the underlying parameters. In some cases the actual parameter values had changed in the course of time; in other cases the determination of the parameters had not been accurate enough to ascertain correct planetary positions over periods of centuries.

Many of the extant manuscripts of zījes contain a mixture of material from various sources. Since these sources are not always explicitly mentioned, it is often difficult to determine the origin of tables in zījes. We can assume that certain mathematical properties of a table, such as the method of computation and the underlying parameter values, are typical for the astronomer who calculated the table. Thus we may be able to identify the origin of a table by investigating its mathematical properties. Usually the tabular headings and the explanatory text in zījes provide little information about the method of computation and the underlying parameter values of the tables. Therefore such information must be extracted from the tabular values themselves. In order to determine the unknown parameter values as reliably as possible and to find important details of the methods of computation, the use of advanced mathematical and statistical methods turns out to be both essential and very effective, as is shown convincingly in van Dalen 1993 and in Van Brummelen 1993.

In this article I will show how mathematical and statistical methods can be used to determine the tabulated function, the underlying parameter values and the author of a table about which no textual information is available. In the analysis of the table I have as much as possible left out the statistical details. A summary of the

1 An overview of all zījes known in 1956 and of the types of tables that occur in zījes can be found in Kennedy 1956.

statistical methods used can be found in the Appendix at the end of this article; for more extensive explanations the reader is referred to my doctoral thesis.

Abu'l-Ḥasan Kūshyār ibn Labbān ibn Bāshahrī al-Jīlī worked as an astronomer in Baghdad around the year 1000. The attribute al-Jīlī indicates that Kūshyār was a native of the region Jīlān in northern Iran. Kūshyār's main achievements were in the fields of arithmetic, trigonometry and astronomy. He wrote a work "The Elements of Hindu Reckoning" about sexagesimal arithmetic and computed extensive trigonometric tables. In his astronomical works Kūshyār made use of the parameters of al-Battānī (c. 900) instead of making his own observations.[2]

It is unclear whether Kūshyār wrote one or two astronomical handbooks. In "The Book of the Astrolabe" he mentions the Jāmi^c Zīj ("Comprehensive Astronomical Tables") and the Bāligh Zīj ("Extensive Astronomical Tables") as two different works. Kennedy suggests that the Bāligh Zīj is an abridged version of the Jāmi^c.[3] I made a cursory analysis of the tables in four manuscripts of Kūshyār's zīj(es): Istanbul Fatih 3418, Berlin Ahlwardt 5751, Leiden Or. 8 (1054), and Cairo Dār al-Kutub Mīqāt 188/2.[4] The oldest of these manuscripts, Fatih 3418, is entitled "The Book of the Jāmi^c Zīj" and is divided into four treatises containing instructions, tables, explanations and proofs respectively. The same division is found in the Berlin and Leiden manuscripts, although the third and fourth treatises are not actually present in the Berlin manuscript. From the given tables of contents and from the coherence of the material in the Istanbul, Berlin and Leiden manuscripts, it can be concluded that, except for the appended tables described below, both explanatory text and tables in the three manuscripts were part of the original zīj written by Kūshyār.[5] The Cairo manuscript contains only a number of Kūshyār's tables.

My analysis revealed that all four manuscripts contain essentially the same set of somewhat more than 50 tables that are listed in the tables of contents referred to in footnote 5. There are, however, small differences between the manuscripts, which may be due to the existence of two different zījes by the hand of Kūshyār ibn Labbān. As far as the date of compilation of the Jāmi^c Zīj is concerned, Kūshyār gives his planetary apogee values for the year 962, whence it seems

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2 More information about Kūshyār ibn Labbān can be found in the article "Kūshyār" in the Dictionary of Scientific Biography (*DSB*).

3 Kennedy 1956, p. 125 (nos 7 and 9).

4 The Istanbul manuscript was copied in the year 1150 and seems to contain the Jāmi^c Zīj in its original form. The Berlin manuscript is described in Ahlwardt 1893, pp. 203-206, which also gives an extensive table of contents. The Leiden manuscript is analysed in Kennedy 1956, pp. 156-157. The Cairo manuscript is described in King 1986, p. 45 (no. B70). Sezgin, GAS, vol. 6, pp. 247-248 mentions six more manuscripts that contain fragments of the Jāmi^c Zīj, but these do not contain Kūshyār's tables.

5 The table of contents of the explanatory text can be found in Fatih 3418, folios 1^v-2^v; Berlin Ahlwardt 5751, pp. 2-4 (also given in Ahlwardt 1893, pp. 204-205) and Leiden Or. 8 (1054), folios 1^v-2^v. The list of tables can be found in Fatih 3418, folio 37^v; Berlin Ahlwardt 5751, p. 35 (also given in Ahlwardt 1893, p. 205) and Leiden Or. 8 (1054), folio 21^r.

plausible that he compiled his zīj(es) shortly after this date.[6] This is confirmed by a reference in Sezgin, GAS, which indicates that from one of the manuscripts of the Jāmiᶜ Zīj it can be concluded that Kūshyār finished his zīj in 964.[7]

At the end of the Berlin and Leiden manuscripts of the Jāmiᶜ Zīj we find a large number of tables that apparently were not part of Kūshyār's original work. In many cases these tables display functions that can also be found in the main set of tables. In the Berlin manuscript, a number of planetary equation tables are attributed to Ibn al-Aᶜlam (c. 960), some other tables to Abū Maᶜshar (Albumasar, c. 850). In the Leiden manuscript, a set of planetary equation tables is taken from the Fākhir Zīj by al-Nasawī (c. 1030), some other tables mention al-Bīrūnī (c. 1000) as their author. However, most of the appended tables in both manuscripts are not attributed.

One of the tables in the manuscript Berlin Ahlwardt 5751 which is not part of the main set of tables, occurs on pages 178-179. The first half of this table is entitled "Table of the Solar Equation", the second half "Table of the Equation of the Mean Solar Position". The argument is the mean solar position and tabular values are displayed in zodiacal signs, degrees, minutes and seconds for every degree of the ecliptic. No further information about the tabulated function or the author of the table is found. In this article we will unravel the mathematical structure of this table and we will determine the values of the underlying parameters. All through this article the table will be referred to as "the true solar longitude table in the Jāmiᶜ Zīj".

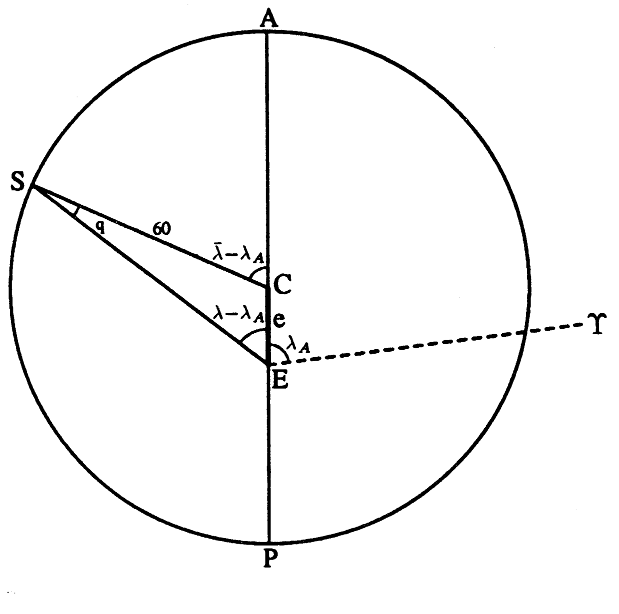

Figure 1: The Ptolemaic solar model

6 See Istanbul Fatih 3418, folio 45ᵛ.

7 See Sezgin, GAS, vol. 5, pp. 343-344.

Islamic astronomers used the solar model described in Ptolemy's Almagest (see Figure 1).[8] According to this model, the sun S moves at uniform speed along a circle with a radius of 60 units. The centre C of this circle is removed from the earth E by a distance e, the solar eccentricity. The sun reaches its greatest distance from the earth at the apogee A and its least distance at the perigee P. The true solar longitude λ is the position of the sun as seen from the earth and is measured by the angle ΥES, where Υ is the vernal equinoctial point. The angle ΥEA measures the longitude of the apogee and is denoted by λ_A. The true solar longitude differs from the mean solar longitude $\bar{\lambda}$, a linear function of time, by a variable quantity q, called the solar equation. Usually $\bar{\lambda}$ is defined in such a way that the angle ACS is always equal to $\bar{\lambda} - \lambda_A$. Note that we then have $\lambda = \bar{\lambda} = \lambda_A$ when the sun is at its apogee and $\lambda = \bar{\lambda} = \lambda_A + 180°$ when the sun is at its perigee. Furthermore $\lambda < \bar{\lambda}$ when $\lambda_A < \lambda < \lambda_A + 180°$ and $\lambda < \bar{\lambda}$ when $\lambda_A - 180° < \lambda < \lambda_A$. The solar equation can be calculated as a function of the true solar longitude by applying the sine rule to the triangle CES:

$$q(\lambda) = \arcsin\left(\frac{e}{60}\sin(\lambda - \bar{\lambda}_A)\right). \tag{1}$$

The solar equation can be calculated as a function of the mean solar longitude by extending the triangle CES to a right-angled triangle DES in which $EDS = 90°$. Then

$$q(\bar{\lambda}) = \arctan\left(\frac{DE}{DS}\right) = \arctan\left(\frac{e\sin(\bar{\lambda} - \lambda_A)}{60 + e\cos(\bar{\lambda} - \lambda_A)}\right). \tag{2}$$

An inspection of the values in the true solar longitude table in the Jāmi^c Zīj reveals that what has been tabulated is the true solar longitude λ as a function of the mean solar longitude $\bar{\lambda}$. In fact, the difference between tabular value and argument is roughly a sinusoidal function which never exceeds 2 in absolute value, is negative from 23° Gemini to 22° Sagittarius and positive otherwise. Thus we expect that the tabulated function will be

$$\lambda(\bar{\lambda}) = \bar{\lambda} - q(\bar{\lambda}) = \bar{\lambda} - \arctan\left(\frac{e\sin(\bar{\lambda} - \lambda_A)}{60 + e\cos(\bar{\lambda} - \lambda_A)}\right), \tag{3}$$

where q denotes the solar equation, e is the solar eccentricity and λ_A the longitude of the solar apogee.

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8 For extensive descriptions of the Ptolemaic solar model, see Pedersen 1974, pp. 122-158 or Toomer 1984, pp. 131-172. For exact and approximative methods that were used by mediaeval Islamic astronomers to compute the solar equation, see Kennedy 1977 and Kennedy & Muruwwa 1958.

$\bar\lambda$	$T(\bar\lambda)$	$D^{(1)}(\bar\lambda)$	$\bar\lambda$	$T(\bar\lambda)$	$D^{(1)}(\bar\lambda)$	$\bar\lambda$	$T(\bar\lambda)$	$D^{(1)}(\bar\lambda)$
300	301;11,24	1; 1,37	320	321;40, 3	1; 1,10	340	341;56,49	1; 0, 4
301	302;13, 1	1; 1,37	321	322;41,13	1; 1, 9	341	342;56,53	1; 0,44
302	303;14,38	1; 1,37	322	323;42,22	1; 1, 9	342	343;57,37	1; 0,24
303	304;16,15	1;1,36	323	324;43,31	1; 1, 9	343	344;58, 1	1; 0,34
304	305;17,51	1; 1,36	324	325;44,40	1; 1, 9	344	345;58,35	1; 0,13
305	306;19,27	1; 1,32	325	326;45,49	1; 0,51	345	346;58,48	0;59,30
306	307;20,59	1; 1,32	326	327;46,40	1; 0,51	346	347;58,18	1; 0,50
307	308;22,31	1; 1,32	327	328;47,31	1; 0,50	347	348;59, 8	1; 0,10
308	309;24, 3	1; 1,32	328	329;48,21	1; 0,50	348	349;59,18	1; 0,20
309	310;25,35	1; 1,32	329	330;49,11	1; 0,50	349	350;59,38	1; 0,10
310	311;27, 7	1; 1,18	330	331;50, 1	1; 0,47	350	351;59,48	1; 0, 0
311	312;28,25	1; 1,18	331	332;50,48	1; 0,47	351	352;59,48	1; 0, 0
312	313;29,43	1; 1,18	332	333;51,22	1; 0,47	352	353;59,48	1; 0, 0
313	314;31,19	1; 1,18	333	334;52,22	1; 0,47	353	354;59,48	1; 0, 1
314	315;32,19	1; 1,18	334	335;53, 9	1; 0,47	354	355;59,49	1; 0, 0
315	316;33,37	1; 1,18	335	336;53,56	1; 0,35	355	356;59,49	0;59,49
316	317;34,55	1; 1,17	336	337;54,31	1; 0,35	356	357;59,38	0;59,49
317	318;36,12	1; 1,17	337	338;55, 6	1; 0,35	357	358;59,27	0;59,49
318	319;37,29	1; 1,17	338	339;55,41	1; 0,35	358	359;59,16	0;59,49
319	320;38,46	1; 1,17	339	340;56,16	1; 0,33	359	360;59, 5	0;59,49
						360	361;58,54	

Table 1: Tabular differences of Kūshyār's true solar longitude table

k	$\hat{a}_k$	$\hat{b}_k$
0	-0.0000501543	
1	-1.9684899033	0.2530317106
2	-0.0002099123	-0.0004847958
3	-0.0131708496	0.0049899528
4	-0.0001228737	-0.0003738902
5	0.0004548023	-0.0002402755
6	0.0001358618	0.0002674402
7	0.0006174986	-0.0000334333
8	0.0002106039	0.0005374500
9	-0.0001912837	0.0002584406
10	-0.0000278764	0.0003071745

Table 2: Fourier coefficients of the reconstructed solar equation

Table 1 displays some of the tabular values $T(\overline{\lambda})$ and their first order differences $D^{(1)}(\overline{\lambda}) \stackrel{\text{def}}{=} T(\overline{\lambda}+1) - T(\overline{\lambda})$.[9] It can be seen at once that linear interpolation within intervals of 5 degrees of the argument was applied. Moreover, it seems probable that what I will call "distributed linear interpolation" was used: the tabular differences are distributed over the intervals of 5 degrees in such a way that they are increasing or decreasing over as long as possible stretches of the argument.[10] After the obvious scribal errors indicated below have been corrected, 57 out of the 72 intervals of 5 degrees are in agreement with the assumption that distributed linear interpolation was used. In addition to the intervals on which the tabular differences are constant, only one interval is in agreement with the possibility of ordinary interpolation. By means of the irregularities in the first order differences, a number of obvious scribal errors in the tabular values can be corrected. For example, the irregularities in the differences for arguments 340 to 346 can be removed by correcting $T(341)$ to 342;57,13, $T(344)$ to 345;58,25 and $T(346)$ to 347;58,58. The correction $T(339)$=340;56,15, which restores the distributed linear interpolation pattern for arguments 335 to 340, is somewhat less plausible. All corrections made in this way are listed in the Apparatus at the end of this article. For the following analysis of the true solar longitude table we will only make use of the independently calculated tabular values.

To the corrected tabular values for multiples of 5 degrees I applied a least squares estimation as explained in the Appendix, based on the assumption that the tabulated function is given by equation (3). The results were as follows:

parameter	*95 % confidence interval*
solar eccentricity	⟨ 2; 4, 4 , 2; 5 ,11⟩
solar apogee	⟨82;25, 6 , 82;55,57⟩

Even though the 95 % confidence intervals contain historically plausible values of the underlying parameters, we must conclude both from the minimum obtainable standard deviation of the tabular errors (1′38′′) and from the sinusoidal error pattern in recomputations for parameter values within the confidence intervals, that the table was not computed according to equation (3).[11] In order to obtain more information about the tabulated function we will make use of Fourier coefficients.

Let $T_q(\overline{\lambda}) \stackrel{\text{def}}{=} \overline{\lambda} - T(\overline{\lambda})$ denote the solar equation table that can be reconstructed from the true solar longitude values in the Jāmi^c Zīj by subtracting them from the

9 Practically all astronomical tables in ancient and mediaeval sources display values in sexagesimal notation. In transcribing sexagesimal numbers we will follow the convention that sexagesimal digits are separated by a comma and that the sexagesimal point is indicated by a semicolon. Thus the sexagesimal number 1;59,56 denotes $1 \cdot 60^0 + 59 \cdot 60^{-1} + 56 \cdot 60^{-2}$.

10 In the case of ordinary linear interpolation the differences are distributed evenly between every two independently calculated values. Thus the tabular differences for arguments 300 to 305 would have been 1;1,37, 1;1,36, 1;1,37, 1;1,36 and 1;1,37 respectively (cf. Table 1).

11 For a table with accurate values to seconds, the minimum obtainable standard deviation of the tabular errors is approximately 17′′′. This follows from the fact that the distribution of the tabular errors can be shown to be approximately uniform; see van Dalen 1993, Section 1.2.4.

mean solar longitude. $T_q(\overline{\lambda})$, $\overline{\lambda} = 0,5,10,\ldots,355$ are then tabular values for a function f which satisfies $f(x) = g(x - \lambda_A)$ for every x, where g is an odd function with period 360°.[12] Thus we can use the approximated Fourier coefficients described in the Appendix.

Table 2 displays the approximated Fourier coefficients

$$\hat{a}_k \overset{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(5i)\cos 5ik$$

for $k = 0,1,2,\ldots,10$ and

$$\hat{b}_k \overset{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(5i)\sin 5ik$$

for $k=1,2,3,\ldots 10$.[13] We can note the following:

- The Fourier coefficients converge rapidly, as can be seen from $\hat{a}_1$, $\hat{a}_3$, $\hat{a}_5$ and $\hat{b}_1$, $\hat{b}_3$, $\hat{b}_5$.
- The coefficients $\hat{a}_2$ and $\hat{a}_4$ are significantly smaller than $\hat{a}_1$ and $\hat{a}_3$. Similarly, $\hat{b}_2$ and $\hat{b}_4$ are significantly smaller than $\hat{b}_1$ and $\hat{b}_3$. In the Appendix it is shown that if the odd, periodic function g as introduced above satisfies the symmetry relation $g(180 - x) = g(x)$ for every x, then all Fourier coefficients $\hat{a}_k$ and $\hat{b}_k$ for even k are zero. We conclude that the function f tabulated in Kūshyār's true solar longitude table is such that g contains this symmetry. If the symmetry were present in all tabular values, the approximated Fourier coefficients $\hat{a}_k$ and $\hat{b}_k$ for even k would actually be zero. The fact that they are small but non-zero is a result of scribal and computational errors that will be discovered (and partially corrected) below.
- All approximated Fourier coefficients contain random errors that derive from the rounding errors (and partially from scribal and computational errors) in the tabular values. In the approximated coefficients $\hat{a}_1$, $\hat{a}_3$, $\hat{b}_1$ and $\hat{b}_3$ these errors cannot be recognized, since they are smaller than the Fourier coefficients themselves. In all remaining approximated coefficients the errors overwhelme the actual Fourier coefficients.

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12 The solar equation always has this property, regardless of which of the three common methods was used for its computation (cf. Kennedy 1977). Furthermore it can be checked directly that the values $T_q(\overline{\lambda})$ have this property at least approximately.

13 The approximated coefficients for $k > 10$ are not displayed, since they show the same behaviour as the coefficients for $k=4$ to 10.

$\bar{\lambda}$	$T_q(\bar{\lambda})$	$T_q(\bar{\lambda})$-$f_3(\bar{\lambda})$	$\bar{\lambda}$	$T_q(\bar{\lambda})$	$T_q(\bar{\lambda})$-$f_3(\bar{\lambda})$
0	-1;58,54	+1´´	180	1;58,54	+1´´
5	-1;56,59	+3	185	1;56,59	-1
10	-1;54,10	+4	190	1;54,10	-1
15	-1;50,27	+5	195	1;50,27	-2
* 20	-1;46, 0	-2	* 200	1;46, 0	+6
25	-1;40,27	+7	205	1;40,27	-4
* 30	-1;34,59	-33	210	1;34,18	-4
* 35	-1;27,44	-9	* 215	1;27,44	+13
40	-1;19,58	+8	220	1;19,58	-4
45	-1;11,56	+6	225	1;11,54	-5
50	-1; 3,24	+5	230	1; 3,24	-2
55	-0;54,24	+6	235	0;54,25	-2
60	-0;45, 6	+2	240	0;45, 9	+3
65	-0;35,27	+2	245	0;35,27	
70	-0;25,37	-1	250	0;25,37	+2
75	-0;15,33	+1	255	0;15,34	+1
80	-0; 5,25		260	0; 5,26	+1
85	0; 4,44	-1	265	-0; 4,44	
90	0;14,52	-2	270	-0;14,52	
95	0;24,56		275	-0;24,55	
100	0;34,48	-2	280	-0;34,48	-1
105	0;44,27	-2	285	-0;44,29	-3
110	0;53,51		290	-0;53,49	-1
115	1; 2,49	-3	295	-1; 2,49	-1
120	1;11,23	-4	300	-1;11,24	-1
125	1;19,27	-5	305	-1;19,27	+1
* 130	1;27, 7	+4	* 310	-1;27, 7	-8
135	1;33,52	-4	* 315	-1;33,37	+16
140	1;40, 3	-4	320	-1;40, 3	+1
* 145	1;46, 1	+27	* 325	-1;45,49	-18
150	1;50,10	-2	* 330	-1;50, 1	+9
155	1;53,56	-2	335	-1;53,56	
160	1;56,50		340	-1;56,49	+1
165	1;58,47		345	-1;58,48	-1
* 170	1;59,36	-11	350	-1;59,48	-1
175	1;59,49		355	-1;59,49	+1

Table 3: Differences between the reconstructed solar equation and an approximation based on approximated Fourier coefficients

We can now compare tabular values $T_q(5i)$ and $T_q(5i+180)$ to see whether the symmetry $f(x) = -f(x+180)$, which follows from $g(x) = -g(-x)$ combined with $g(180-x) = g(x)$, is in fact present in the reconstructed solar equation table. Furthermore, we can compare the tabular values with approximated functional values $f_K(x)$ obtained from a finite Fourier series based on the approximated coefficients:

$$f_K(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\hat{a}_0 + \sum_{k=1}^{K} (\hat{a}_k \cos kx + \hat{b}_k \sin kx) \qquad (4)$$

Since in this case all approximated Fourier coefficients starting from $k=4$ are of the same order of magnitude, we choose $K=3$.

Table 3 shows the reconstructed solar equation values $T_q(5i)$ together with the differences (in seconds) between these values and the approximated functional values $f_3(5i)$. It can be noted that the general error pattern is regular, but that there are many outliers, in particular for arguments 20, 30, 35, 130, 145, 170, 200, 215, 310, 315, 325 and 330 (indicated in Table 3 with an asterisk). The tabular values show the expected symmetry $T_q(5i) = -T_q(5i+180)$ in 20 out of 36 cases. For the pairs of arguments 30/210, 135/315, 145/325, 150/330 and 170/350, $T_q(5i)$ differs by more than 8 seconds from $-T_q(5i+180)$. For eleven other pairs we find a small deviation from the expected symmetry.

It turns out that we can find plausible corrections for four of the outliers. The following list gives the reconstructed solar equation values and the corrected values.[14]

reconstructed value	*corrected value*
$T(30)$ = −1;34,59	−1;34,19
$T(170)$ = 1;59,36	1;59,47
$T(315)$ = −1;33,37	−1;33,52
$T(330)$ = −1;50, 1	−1;50,11

Note that in all four cases the correction (approximately) restores both the surrounding error pattern and the symmetry $T_q(5i) = -T_q(5i+180)$. Furthermore all four errors are plausible scribal errors. The other eight outliers occur in pairs of tabular values $T_q(5i)/T_q(5i+180)$ and can therefore not be corrected on the basis of the symmetry of the reconstructed solar equation. Since no plausible corrections on the basis of possible scribal mistakes can be given either, we will leave these outliers unchanged for the time being.

Since the solar equation is an almost linear function in the neighbourhood of its zeros, inverse linear interpolation between $T_q(80)$ and $T_q(85)$ or $T_q(260)$ and $T_q(265)$

14 I have also considered the possibility that the original tabular values for the true solar longitude contain more scribal errors than those that we have corrected on the basis of the interpolation pattern. For instance, the correction of $T_q(170)$ indicated in the list corresponds to a correction of the original true solar longitude value 168;0,24 to 168;0,13.

should ordinarily lead to reasonably accurate values for the solar apogee (in this case the results are $\lambda_A \approx 82;40,6$ and $\lambda_A \approx 82;40,20$ respectively). However, because of the large number of errors in the reconstructed solar equation values, it seems advisable to compute a more accurate estimate and a confidence interval for λ_A based on all tabular values by means of the Fourier estimator explained in the Appendix. In this way we arrive at an estimate

$$\hat{\lambda}_A \stackrel{\text{def}}{=} \arctan(\hat{a}_1 / \hat{b}_1) = 82;40,1$$

for the apogee. Using

$$\sigma^2 \approx \frac{1}{n-1} \sum_{i=1}^{n} (T_q(5i) - f_3(5i))^2, \tag{5}$$

with f_3 according to equation (4), it follows that the standard deviation σ of the tabular errors is approximately equal to 4″51‴. Assuming that the distribution of the Fourier estimator is approximately normal we find ⟨82;38,56 , 82;41,6⟩ as an approximate 95 % confidence interval for the solar apogee.

Seeing that none of the remaining outliers occurs for a multiple of 15°, we can obtain a better estimate and a smaller approximate 95 % confidence interval for the solar apogee by applying the Fourier estimator to the set of solar equation values $T_q(15i)$, $i = 1,2,3,\ldots,24$. It turns out that we can then use $K=5$ in our approximation (4) of the functional values, since now $\hat{a}_5$ and $\hat{b}_5$ are significantly larger than the approximated Fourier coefficients with larger indices. The resulting estimate for the solar apogee is $\hat{\lambda}_A=82;40,6$, the approximated standard deviation of the tabular errors 52‴47^{iv}. An approximate 95 % confidence interval for the apogee is now found as ⟨82;39,46 , 82;40,26⟩ . Since the differences $T_q(15i)$ - $f_5(15i)$ in fact do not show any outliers, we conclude that the table for the true solar longitude in the Jāmiᶜ Zīj was very probably computed on the basis of the round solar apogee value $\lambda_A = 82°40'$. Below I will explain why this value is also historically plausible.

Because of the symmetry $g(180 - x) = g(x)$ discovered above, we know that the solar equation assumes a maximum for $\lambda_A + 90°$, and a minimum for $\lambda_A + 270°$. By means of third order interpolation between the reconstructed values $T_q(165)$, $T_q(170)$, $T_q(175)$ and $T_q(180)$, we find that the maximum is approximately equal to 1;59,55,13. Similarly, we find that the minimum is close to -1;59,55,41. We conclude that the reconstructed solar equation is probably based on a value of q_{max} close to 1°59′55″ or 1°59′56″ (or, equivalently, on a value of the solar eccentricity e in the neighbourhood of 2°5′34″). Of these values only the maximum solar equation value q_{max} = 1°59′56″ is attested; it occurs in a solar equation table in the Ashrafī Zīj which is attributed to Yaḥyā ibn Abī Manṣūr, one of the astronomers who worked at the court in Baghdad around the year 830.[15] Kennedy found that this solar equation table, which has the mean solar anomaly as its independent variable, was computed according to the so-called "method of declinations", which is probably of Sasanian or early-Islamic origin.[16] We will investigate whether the same holds for the true solar longitude table in the Jāmi^c Zīj, i.e. whether the tabulated function is

$$f(x) = q_{\max} \cdot \frac{\arcsin(\sin(x - \lambda_A) \cdot \sin \varepsilon)}{\varepsilon}. \tag{6}$$

Note that this function satisfies both symmetry relations $f(x-\lambda_A) = -f(-x-\lambda_A)$ and $f(180-x-\lambda_A) = f(x-\lambda_A)$ that we have also found in the table in the Jāmi^c Zīj.

Disregarding the outliers for arguments 20, 35, 130, 145, 200, 215, 310 and 325, which could not plausibly be corrected, I performed a least squares estimation as explained in the Appendix. Assuming that the true solar longitude table in the Jāmi^c Zīj was computed according to the method of declinations, I found that the minimum obtainable standard deviation of the tabular errors is 60´´´, and I obtained the following approximate 95 % confidence intervals:

parameter	*95 % confidence interval*
maximum solar equation	⟨ 1;59,55,18, 1;59,55,47⟩
obliquity of the ecliptic	⟨23;39,15 , 23;48,21 ⟩
solar apogee	⟨82;39,53 , 82;40,13 ⟩

Since the minimum obtainable standard deviation is much higher if we assume any other plausible method of computation,[17] we conclude that the table was very probably computed according to the method of declinations. The confidence interval

15 The Ashrafī Zīj was written in Persian by Muḥammad Sanjar al-Kamālī (Shiraz, south-western Persia, c. 1300). It gives the mean motion parameters and planetary equations from a large number of earlier zījes and is extant in Paris Bibliothèque Nationale Ms. suppl. persan 1488 (288 folios, 1303 A.D.). For more information on the Ashrafī Zīj, see Kennedy 1956, p. 124, no. 4; and Kennedy 1977, p. 183. The solar equation table attributed to Yaḥyā ibn Abī Manṣūr can be found on folio 236^r of the Paris manuscript.

16 See Kennedy 1977.

17 We have already seen that the minimum obtainable standard deviation of the tabular errors is 1´38´´ if we assume that the correct formula

$$q(\bar{\lambda}) = \arctan\left(\frac{e\sin(\bar{\lambda} - \lambda_A)}{60 + e\cos(\bar{\lambda} - \lambda_A)}\right)$$

for the solar equation was applied. Assuming the formula

$$q(\lambda) = \arcsin\left(\frac{e}{60}\sin(\lambda - \lambda_A)\right)$$

for the solar equation as a function of the true solar longitude or the "method of sines"

$$q(\bar{\lambda}) = q_{\max}\sin(\bar{\lambda} - \lambda_A),$$

the minimum obtainable standard deviation is 38´´.

for the solar apogee confirms the results that we have found before by means of the Fourier estimator, so indeed $\lambda_A = 82°40'$. Since the method of declinations is only attested for the Ptolemaic obliquity value,[18] we can assume that $\epsilon = 23°51'$. Fixing these two parameter values, we obtain ⟨1;59,55,27 , 1;59,56,12⟩ as an approximate 95 % confidence interval for the maximum solar equation q_{max}. Thus we see that the true solar longitude table in the Jāmi^c^ Zīj was probably computed using $q_{max} = 1°59'56$.

Conclusion: The table for the true solar longitude which is found on pages 178-179 of the manuscript Berlin Ahlwardt 5751 of Kūshyār ibn Labbān's Jāmi^c^ Zīj was computed according to the so-called "method of declinations" (formula 4). The underlying parameter values are 1°59′56″ for the maximum solar equation, 23°51′ for the obliquity of the ecliptic and 82°40′ for the solar apogee.

I will now argue that all three underlying parameter values are historically plausible and that the true solar longitude table in the Jāmi^c^ Zīj probably derives from Yaḥyā ibn Abī Manṣūr. We have already seen that 1°59′56″ is the maximum solar equation value underlying Yaḥyā's solar equation table in the Ashrafī Zīj. Furthermore, we have seen that the method of declinations is only attested with the value 23°51′ of the obliquity of the ecliptic. I conjecture that the solar apogee value 82°40′ is a rounded version of the value 82°39′, which, according to Ibn Yūnus, was observed at Baghdad in the year 214 Hijra by a group of astronomers headed by Yaḥyā ibn Abī Manṣūr.[19] The solar equation tables in Yahyā's Mumtaḥan Zīj and in the contemporary zīj by Ḥabash al-Ḥāsib extant in Istanbul Yeni Cami 784/2, both indicate that the solar apogee is in 82°39′.[20] The two tables are very probably related, since the first 90 values are practically identical. Ḥabash's zīj contains another table based on the same solar equation values, which displays λ_A plus the solar equation. Here the apogee is taken equal to 82°40′.[21]

We have seen that the true solar longitude table in the Jāmi^c^ Zīj was computed by means of what I call "distributed linear interpolation". The extant recension of the Mumtaḥan Zīj contains a table for the normed right ascension, which is based on obliquity 23°51′ and involves the same type of interpolation.[22] Although this table may simply have been copied from Ptolemy's Handy Tables,[23] it seems

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18 See Kennedy & Muruwwa 1958, p. 118; Kennedy 1977; and Suter 1914, pp. 132-137.

19 See Caussin de Perceval 1804, p. 56 (p. 40 in the separatum).

20 For the table in the Mumtaḥan Zīj, see Escorial Ms. árabe 927, folio 15^r^ or Yaḥyā ibn Abī Manṣūr, p. 28. For the table in Ḥabash's zīj, see Istanbul Yeni Cami 784/2, folios 90^r^-91^r^ and Debarnot 1987, pp. 41-42. Both tables were analysed in Salam & Kennedy 1967, pp. 494-495. The solar equation table in the Mumtaḥan Zīj is completely different from the table in the Ashrafī Zīj attributed to Yaḥyā.

21 See Istanbul Yeni Cami 784/2, folios 200^v^-203^r^ and Debarnot 1987, p. 58.

22 Escorial Ms. árabe 927, folios 48^v^-49^r^ or Yaḥyā ibn Abī Manṣūr, pp. 93-94. See Neugebauer 1975, vol. 1, pp. 31-32 and 42 for more information about the normed right ascension.

23 The normed right ascension table in the Handy Tables can for instance be found in the manuscript Leiden BPG 78, folios 75^r^-76^v^. The table is transcribed in Stahlman 1959, pp. 206-209. The normed right ascension in the Mumtaḥan Zīj is practically identical to the table in

probable that it was an original part of Yaḥyā ibn Abī Manṣūr's zīj, and hence that Yaḥyā was familiar with distributed linear interpolation.

Finally, it can be noted that among the appended tables in the manuscript Berlin Ahlwardt 5751 of Kūshyār ibn Labbān's Jāmi^c Zīj, we find two tables displaying mean planetary positions at two different epochs according to four astronomers, one of them being Yaḥyā ibn Abī Manṣūr.[24] Thus the compiler of the manuscript apparently had access to Yaḥyā's zīj.

We conclude that there is sufficient reason to believe that the true solar longitude table analysed here, like the solar equation table on folio 236^r of the Ashrafī Zīj, originates from Yaḥyā ibn Abī Manṣūr. It seems possible that Yaḥyā's solar equation table as found in the Ashrafī Zīj, was originally contained in the Mumtaḥan Zīj, but was later considered unsatisfactory because of its symmetry (and possibly because of its uncommon value of the maximum equation). Thus we can imagine how in a later recension, like the one that we find in the manuscript Escorial árabe 927, it was replaced, possibly by Ḥabash's table for the solar equation.

Table 4 displays my final recomputation of the solar equation reconstructed from the true solar longitude table in the Jāmi^c Zīj. The second and fifth columns contain the reconstructed solar equation values, the third and sixth columns the differences (in seconds) between these values and a recomputation according to formula (6) using the parameter values found above. Apart from the eight outliers (which are again indicated by an asterisk), the number of differences is 40 out of 64 tabular values, the standard deviation of the differences is 1′′5′′′.

It seems probable that the true solar longitude table in the Jāmi^c Zīj was computed by means of interpolation in a solar equation table like the one in the Ashrafī Zīj. In fact, if linear interpolation were used, the remaining eight outliers in our table could be explained from only two erroneous solar equation values. To see this, we denote the values for the method of declination that were used for the linear interpolation by $q_\delta(\bar{a})$, where $\bar{a} = 1,2,3,\ldots,90$ is the mean solar anomaly. Remembering that $q_\delta(-\bar{a}) = -q_\delta(\bar{a})$ and $q_\delta(180 - \bar{a}) = q_\delta(\bar{a})$, it follows that $T_q(35)$ and hence $-T_q(215)$ were calculated as $\frac{1}{3}q_\delta(47) - \frac{2}{3}q_\delta(48)$, and $T_q(130)$ and $-T_q(310)$ as $\frac{2}{3}q_\delta(47) + \frac{1}{3}q_\delta(48)$. The solar equation values $q_\delta(47) = 1;26,30$ (equal to the value given in the Ashrafī Zīj) and $q_\delta(48) = 1;28,21$ (computational error for 1;27,56?) thus precisely reproduce the four outliers for arguments 35, 130, 215 and 310. In the same way three of the remaining four outliers can be explained if we assume an erroneous value $q_\delta(62) = 1;45,38$ (possible scribal error for the Ashrafī value 1;45,13).

I recomputed the true solar longitude table in the Jāmi^c Zīj by using linear interpolation in Yaḥyā ibn Abī Manṣūr' solar equation table in the Ashrafī Zīj. Disregarding the eight outliers, I found 28 differences in 64 values (as compared to 40 differences for the precise recomputation); the standard deviation of the

the Handy Tables.

24 See Berlin Ahlwardt 5751, pp. 160-161.

differences was $1''3'''$. This result is not good enough to conclude that in fact linear interpolation in the Ashrafī table was applied.

λ	$T_q(\lambda)$	error	λ	$T_q(\lambda)$	error
0	-1;58,54		180	1;58,54	
5	-1;56,59	+1´´	185	1;56,59	-1´´
10	-1;54,10		190	1;54,10	
15	-1;50,27		195	1;50,27	
* 20	-1;46, 0	-9	* 200	1;46, 0	+9
25	-1;40,27	+1	205	1;40,27	-1
30	-1;34,19	-1	210	1;34,18	
* 35	-1;27,44	-17	* 215	1;27,44	+17
40	-1;19,58	+1	220	1;19,58	-1
45	-1;11,56		225	1;11,54	-2
50	-1; 3,24	-1	230	1; 3,24	+1
55	-0;54,24	+1	235	0;54,25	
60	-0;45, 6	-1	240	0;45, 9	+4
65	-0;35,27		245	0;35,27	
70	-0;25,37	-2	250	0;25,37	+2
75	-0;15,33		255	0;15,34	+1
80	-0; 5,25		260	0; 5,26	+1
85	0; 4,44	-1	265	-0; 4,44	+1
90	0;14,52	-1	270	-0;14,52	+1
95	0;24,56	+1	275	-0;24,55	
100	0;34,48		280	-0;34,48	
105	0;44,27		285	-0;44,29	-2
110	0;53,51	+3	290	-0;53,49	-1
115	1; 2,49	+1	295	-1; 2,49	-1
120	1;11,23		300	-1;11,24	-1
125	1;19,27	-1	305	-1;19,27	+1
* 130	1;27, 7	+8	* 310	-1;27, 7	-8
135	1;33,52		315	-1;33,52	
140	1;40, 3	-1	320	-1;40, 3	+1
* 145	1;46, 1	+30	* 325	-1;45,49	-18
150	1;50,10		330	-1;50,11	-1
155	1;53,56	-1	335	-1;53,56	+1
160	1;56,50		340	-1;56,49	+1
165	1;58,47	-1	345	-1;58,48	
170	1;59,47	-1	350	-1;59,48	
175	1;59,49	-1	355	-1;59,49	+1

Table 4: Final recomputation of the reconstructed solar equation

Apparatus

Scribal errors in Kūshyār ibn Labbān's table for the true solar longitude corrected on the basis of the interpolation pattern (the corrected digits are given between brackets; possible errors of 1´´ for arguments 204, 206 and 212 were not corrected):

$\bar{\lambda}$ = 21	$T(\bar{\lambda})$ = 22;44,33	(53´´)	$\bar{\lambda}$ = 136	$T(\bar{\lambda})$ = 134;24,13	(53´´)
33	34;30,18	(38´´)	146	144;14,59	(13´)
47	48; 8,22	(32´´)	147	145;12,59	(19´´)
57	57;50,40	(42´´)	181	179; 1,21	(29´´)
78	78; 9,39	(29´´)	279	279;32,30	(50´´)
104	103;17,18	(28´´)	298	299; 7,18	(58´´)
116	116;15,28	(55´)	341	342;56,53	(57´13´´)
117	115;13,45	(53´)	344	345;58,35	(25´´)
130	128;32,13	(53´´)	346	347;58,18	(58´´)

Appendix

This appendix briefly describes the statistical estimators that I use in this article to determine the mathematical structure and unknown parameter values of Kūshyār ibn Labbān's table for the true solar longitude. More detailed information concerning approximated Fourier coefficients in general and the Fourier estimator in particular can be found in Section 2.3 of my doctoral thesis (van Dalen 1993). A more extensive discussion of least squares estimation can be found in Section 2.4 of my thesis.

The following notations are used throughout this article:
A table for a mathematical function is denoted by T, the tabular values of that table by $T(x)$, where x indicates the argument. The tabulated function is always denoted by f. The tabular errors $e(x)$ are defined by $e(x) = T(x) - f(x)$ for every argument x. Since in this way the tabular error includes the error made by rounding a calculated functional value to the accuracy of the table under consideration, it follows that practically every tabular value contains a non-zero tabular error. An extensive discussion of the distribution of tabular errors can be found in van Dalen 1993, Section 1.2.4. For every parameter that I estimate a so-called 95 % confidence interval is calculated. Such intervals are expected to contain the underlying parameter value in 19 out of 20 cases.

Approximated Fourier coefficients
Let f be a 2π-periodic function and assume that the Fourier series of f converges, i.e. that, for every $x \in [0,2\pi]$,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx$$

and

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx$$

for every k.

Now suppose that we have a table T of the function f with tabular values $T(i\alpha)$ for $i = 1,2,3,\ldots,n$, where n is a multiple of 4 and $\alpha = 2\pi/n$. Since for every 2π-periodic, twice continuously differentiable function h we have

$$\int_0^{2\pi} h(x)dx = \frac{2\pi}{n}\sum_{i=1}^{n} h(i\alpha) + \frac{\pi^3}{3n^3}h''(\xi),$$

where $\xi \in [0,2\pi]$, it follows that the Fourier coefficients a_k and b_k of the function f can be approximated by

$$\widetilde{a}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} f(i\alpha)\cos ik\alpha$$

and

$$\widetilde{b}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} f(i\alpha)\sin ik\alpha$$

respectively. I found that in practice the errors in these approximations can be neglected if the number of tabular values n is at least equal to twelve.

Since we do not know the functional values $f(i\alpha)$ themselves, we have to approximate them by the given tabular values $T(i\alpha)$. Thus we will estimate the Fourier coefficients a_k and b_k by

$$\hat{a}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\cos ik\alpha$$

and

$$\hat{b}_k \stackrel{\text{def}}{=} \frac{2}{n}\sum_{i=1}^{n} T(i\alpha)\sin ik\alpha$$

respectively. If the tabular errors $T(i\alpha) - f(i\alpha)$ are denoted by $e(i\alpha)$, then we have

$$\hat{a}_k - \tilde{a}_k = \frac{2}{n}\sum_{i=1}^{n} e(i\alpha)\cos ik\alpha$$

and

$$\hat{b}_k - \tilde{b}_k = \frac{2}{n}\sum_{i=1}^{n} e(i\alpha)\sin ik\alpha.$$

Assuming that the tabular errors are mutually independent and have a common mean 0 and variance σ^2, we find that $\hat{a}_k$ and $\hat{b}_k$ have a negligible bias (namely the error made in the approximation of the integral in the definition of the Fourier coefficients by a finite sum) and a variance Var $\hat{a}_k$ = Var $\hat{b}_k = 2\sigma^2/n$. Furthermore $\hat{a}_k$ and $\hat{b}_k$ have a zero covariance. Using the Central Limit Theorem we find that both $\hat{a}_k$ and $\hat{b}_k$ have a distribution which is approximately normal.

In Islamic astronomical handbooks we find tables of 2π-periodic functions f that, for some odd function g and a constant λ_A, satisfy the relation $f(x) = g(x-\lambda_A)$ for every x.[25] If the Fourier series of f converges, the Fourier series of g converges as well, and we can derive the Fourier coefficients of g from those of f. For every $x \in [0,2\pi]$ we have

$$\begin{aligned}
g(x) &= f(x+\lambda_A) \\
&= \tfrac{1}{2}a_0 \\
&\quad + \sum_{k=1}^{\infty} a_k(\cos kx \cdot \cos k\lambda_A - \sin kx \cdot \sin k\lambda_A) \\
&\quad + \sum_{k=1}^{\infty} b_k(\sin kx \cdot \cos k\lambda_A + \cos kx \cdot \sin k\lambda_A) \\
&= \tfrac{1}{2}a_0 \\
&\quad + \sum_{k=1}^{\infty} (a_k \cos k\lambda_A + b_k \sin k\lambda_A)\cos kx \\
&\quad + \sum_{k=1}^{\infty} (b_k \cos k\lambda_A - a_k \sin k\lambda_A)\sin kx.
\end{aligned}$$

Since g is an odd function, it follows that for every k we have $a_k\cos k\lambda_A + b_k\sin k\lambda_A = 0$. In cases where g also satisfies the symmetry relation $g(180-x) = g(x)$ for every x, it can be shown that $f(x+180) = -f(x)$ for every x and that $a_k = b_k = 0$ when k is even. If the tabular values satisfy the symmetry relation $T(x+180) = -T(x)$ for every x, then $\hat{a}_k = \hat{b}_k = 0$ for even k.

25 The notation λ_A is chosen since in practice the constant will often be the solar apogee.

Fourier estimator

As above, let f be a 2π-periodic function with convergent Fourier series. Assume that $f(x) = g(x-\lambda_A)$ for every x, where g is an odd function and λ_A an unknown constant. Let T be a table for f with tabular values $T(i\alpha)$, $i = 1,2,3,\ldots,n$, where n is a multiple of 4 and $\alpha=2\pi/n$. Again let $e(i\alpha)$ denote the tabular errors $T(i\alpha)$ -$f(i\alpha)$ and σ^2 the common variance of these errors. Let a_k and b_k denote the Fourier coefficients of the function f and let $\hat{a}_k$ and $\hat{b}_k$ be the estimators for these coefficients introduced above.

We have seen that a_k and b_k satisfy the relation $a_k \cos k\lambda_A + b_k \sin k\lambda_A = 0$ for every k. Provided that b_k is not equal to zero, this implies that $\tan k\lambda_A = -a_k / b_k$ for every k and that $\hat{\lambda}_A(k)$ defined by $\tan \hat{\lambda}_A(k)$ is an estimator for λ_A for every k. The calculation of the accuracy of this estimator is straightforward and can be found in my doctoral thesis.[26] It turns out that the bias of $\hat{\lambda}_A$ is negligible (it is of the order of σ^3 for $\sigma \rightarrow 0$) and that

$$\operatorname{Var} \hat{\lambda}_A = \frac{180^2}{\pi^2} \frac{2\sigma^2}{nk^2(a_k^2 + b_k^2)} + O(\sigma^3).$$

Since the Fourier coefficients converge rapidly for most functions f of which we find tables in Islamic astronomical handbooks, the estimator with the smallest variance is obtained for $k=1$. I call this estimator the "Fourier estimator". The distribution of the Fourier estimator is in most cases very close to normal.

Least Squares estimation

Let T be a table with tabular values $T(x)$, $x \in X$, for the function f_θ which depends on the parameter vector θ. Let the objective function $\Phi(\theta)$ be defined as the sum of the squares of the tabular errors:

$$\Phi(\theta) = \sum_{x \in x} (t(x) - f_0(x))^2.$$

A least squares estimate for the parameter vector θ is a vector $\hat{\theta}$ that minimizes $\Phi(\theta)$.

Whenever f_θ is a non-linear function, we need an iterative optimization procedure in order to calculate a least squares estimate for the parameter vector underlying a particular table. It turns out that for most types of tables in Islamic astronomical handbooks the so-called Gauss-Newton procedure converges rapidly to a least squares estimate. Once the estimate $\hat{\theta}$ has been obtained, we can approximate the standard deviation σ of the tabular errors from $\sigma^2 \approx \Phi(\hat{\theta})/n$. I refer to this approximation as the "minimum obtainable standard deviation" of the tabular errors. If the minimum obtainable standard deviation is much larger than what we expect on the basis of the number of sexagesimal digits of the tabular values, then

26 See van Dalen 1993, pp. 49-50.

it is probable that f_θ is not the tabulated function. If the minimum obtainable standard deviation is small enough and the tabular errors can be assumed to be independent and to have zero means and equal variances, then separate confidence intervals for all underlying parameters of the table under consideration can be computed from the found least squares estimate. More information about least squares estimation can for instance be found in van Dalen 1993, Section 2.4 or in Draper & Smith 1981, Chapter 10.

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