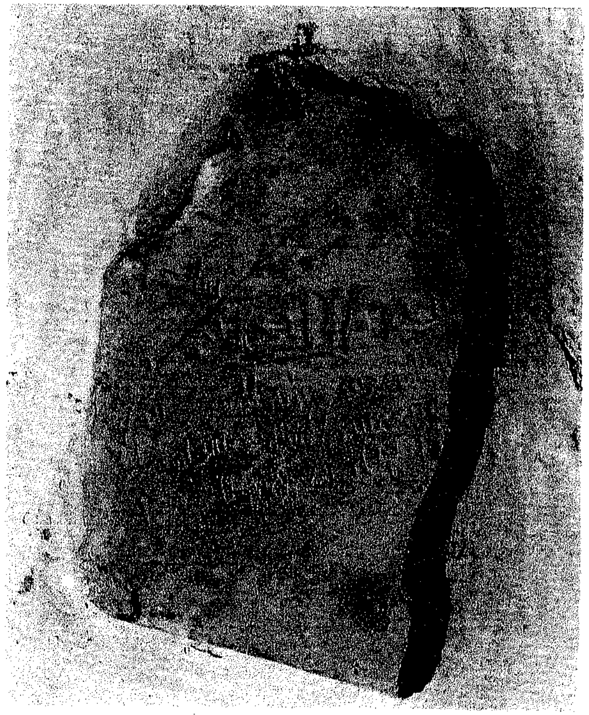
Studies in the Islamic Exact Sciences in Honour of Prof. Juan Vernet



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en honor del Prof. Juan Vernet

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AL-KHWĀRIZMĪ'S ASTRONOMICAL TABLES REVISITED: ANALYSIS OF THE EQUATION OF TIME

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1. *Introduction*

Al-Khwārizmī was an influential Muslim mathematician, astronomer and geographer who lived in Baghdad in the first half of the 9th century A.D. His main astronomical work was a *zīj*, an astronomical handbook with tables and explanatory text, called the *Sindhind Zīj*. This work was largely based on Indian methods, as opposed to most later Islamic astronomical handbooks which utilized the Greek planetary models laid out in Ptolemy's *Almagest*. The *Sindhind Zīj* is only extant in a Latin translation of a recension by Maslama al-Majrīṭī (Cordoba, c. 980). Through this translation and the so-called *Toledan Tables* some of the Indian methods used by al-Khwārizmī made their way to Western Europe.

The mathematical structure and underlying parameter values of practically all tables in the Latin version of al-Khwārizmī's *Sindhind Zīj* have been determined. Using the mathematical information obtained in this way the origin of most of the tables could be ascertained. One of the very few tables of which the mathematical structure has not yet been established, is the table for the equation of time. In this article I will present a full analysis of this table and will show that it is based on two Ptolemaic parameter values plus a value found by the group of astronomers who compiled the *Mumtaḥan Zīj* (Baghdad, c. 830).

The main mathematical tool used for the analysis of al-Khwārizmī's table for the equation of time is the method of least squares. The application of this method will be described step by step. In this way it is hoped to enable the reader to perform similar determinations of the unknown parameters of an astronomical table by means of the computer program TA (Table-Analysis), available from the author.

In Section 2 of this article information is presented concerning al-Khwārizmī's life and works. Section 3 gives an overview of the available primary and secondary sources related to the *Sindhind Zīj*. In Section 4 I present a detailed survey of previous results about al-Khwārizmī's astronomical tables, including the most important technical details and ample references. After the explanation of the equation of time in Section 5, al-Khwārizmī's table for the equation of time will be extensively analysed in Section 6. A summary of the results of this analysis can be found in Section 7.

2. *Al-Khwārizmī's life and works*

Abū Jaᶜfar Muḥammad ibn Mūsā al-Khwārizmī lived in the first half of the 9th century A.D.[2] His name indicates that his ancestors came from Khorezm, a region south of the Aral sea. According to the historian al-Ṭabarī (Baghdad, 839-923 A.D.), al-Khwārizmī himself came from Quṭrubbul, a suburb of Baghdad.

Al-Khwārizmī was active in Baghdad as a mathematician, astronomer and geographer during the reigns of the Abbasid caliphs al-Ma'mūn (813-833), al-Muᶜtaṣim (833-842) and al-Wāthiq (842-847). During the reign of al-Ma'mūn he became a member of the "House of Wisdom", a scientific institution strongly supported by the caliph (cf. the *EI*² article "Bayt al-ḥikma"). Al-Khwārizmī's works on algebra and astronomy were dedicated to al-Ma'mūn and were hence probably written before 833. His treatise on Hindu numerals refers to the work on algebra and must therefore be later; his treatise on the Jewish calendar gives an example for the year 823 / 824. The dating of al-Khwārizmī's remaining known works, a treatise on geography, a chronicle, a treatise on the sundial and two treatises on the astrolabe is problematic.

Al-Khwārizmī's works were influential both in the Arab world and in medieval Europe. His work on algebra *al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala* (*The Compendium on Calculation by Completing and Balancing*) was in use as a textbook for several centuries and served as an archetype for treatises on algebra by later authors. The Latin translation of this work stood at the basis of the development of European algebra, to which it gave its name.

The Latin translation of al-Khwārizmī's work on arithmetic with Hindu numerals, of which the original Arabic text is no longer extant, initiated a number of 12th and 13th-century European works on arithmetic. The titles of many of these works contained the Latinized version "Algorismus" of al-Khwārizmī's name, from which our word "algorithm" derives.

[2] Most of the following information was taken from the *DSB* article "al-Khwārizmī" by Gerald J. Toomer. For more extensive references and further biographical and bibliographical information the reader is also referred to the *EI*² article "al-Khwārazmī" by Juan Vernet, and to Sezgin 1971-1984, vol. 5, pp. 228-241 and vol. 6, pp. 140-143.

Al-Khwārizmī's main astronomical work was called the *Sindhind Zīj*.[3] It was largely based on Indian methods and parameter values taken from the *Sindhind*, an Arabic translation (made around the year 770 by al-Fazārī) of the *Brāhmasphuṭasiddhānta* by the 7th-century Indian astronomer Brahmagupta. Other elements were taken from the *Shāh Zīj*, a non-extant Persian work of the 6th century, and from the *Khaṇḍakhādyaka*, another work by Brahmagupta. The *Sindhind Zīj* existed in a larger version, which included explanations of the models used, and a smaller version containing only tables and instructions for their use. Neither version is extant in the original Arabic. The smaller version became known in Spain in the 9th century and a recension of it was made by the 10th-century Muslim mathematician and astronomer Abū'l-Qāsim Maslama ibn Aḥmad al-Faraḍī al-Majrīṭī, who worked in Cordoba.[4] According to the 11th-century historian and astronomer Ṣāʿid al-Andalusī, al-Majrīṭī converted the planetary tables in al-Khwārizmī's *zīj* from the Persian to the Arabic calendar and adapted some of the tables to the geographical longitude of Cordoba. Al-Majrīṭī's recension is only extant in a 12th-century Latin translation by Adelard of Bath, which is the main source for research on al-Khwārizmī's astronomical tables (cf. the following section).

3. *Sources for the study of al-Khwārizmī's astronomical tables*

For the study of al-Khwārizmī's *Sindhind Zīj* the following

[3] The Arabic word *zīj* derives from the middle Persian *zīg*. It indicates an astronomical handbook with tables and explanatory text.

[4] Al-Majrīṭī is known to have written a work on commercial arithmetic (*Muʿāmalāt*) and was the first Andalusian astronomer who made astronomical observations of his own. His disciples, among whom were Ibn al-Ṣaffār, Ibn al-Samḥ, ʿAmr ibn ʿAbd al-Raḥmān al-Kirmānī and Ibn Barghūth, were influential mathematicians and astronomers throughout Spain. For further information, see the *DSB* article "al-Majrīṭī" and the *EI²* article "al-Madjrīṭī" by Juan Vernet. See also Vernet & Catalá 1965 and Samsó 1992, pp. 80-110.

primary sources are available:[5]

1) *The Latin translation by Adelard of Bath of al-Majrīṭī's recension of the smaller version of al-Khwārizmī's zīj*. This translation is available in nine manuscripts, some of which contain fragments only. The relationship between the manuscripts was discussed extensively in Mercier 1987, which lists the manuscripts in footnote 9 on page 89. The manuscripts Chartres Bibliothèque publique No. 214 (173), Madrid Biblioteca Nacional No. 10016, Oxford Bodleian Library Cod. Auct. F.I. 9 (Bernard No. 4137) and Paris Bibliothèque Mazarine No. 3642 (1258) were used by Suter for his edition and commentary published in 1914. Neugebauer (1962) translated the Latin version of al-Khwārizmī's *zīj* into English and provided a new commentary giving many new insights into the mathematical structure and the origin of the tables. He included a complete edition and translation of the manuscript Oxford Corpus Christi College Ms. 283. Recently Pedersen (1992) established that a set of astronomical rules in the Latin manuscript Oxford Merton College 259 is close to al-Khwārizmī's original *zīj*.
2) *The commentary on the larger version of al-Khwārizmī's zīj by Ibn al-Muthannā*. This 10th-century work is lost in the Arabic original. A Latin translation by Hugo Sanctallensis is available in the manuscripts Oxford Bodleian Library Arch. Selden B 34, Oxford Bodleian Library Savile 15, and Cambridge Gonville and Caius College 456. Two Hebrew translations, one of which by Ibn Ezra, can be found in the manuscripts Parma Biblioteca Palatina 2636 (De Rossi 212) and Oxford Bodleian Library Ms. Michael 400. The Latin translation was edited in Millás Vendrell 1963; the Hebrew versions were edited and translated in Goldstein 1967.
3) *The commentary on al-Khwārizmī's zīj by Ibn Masrūr*. This 10th-century commentary, entitled *Kitāb ʿilal al-zījāt* (*Book of the reasons of the zījes*) and available as Cairo Taymūr Math. 99 (see King 1986, no. B37, p. 38), has not been published. Kennedy and Ukashah consulted the manuscript for their investigation of al-Khwārizmī's tables for planetary latitude (1969), King for his research about lunar

[5] Detailed information about the manuscripts listed below can be found in the secondary sources indicated.

crescent visibility tables (1987).

4) *The Toledan Tables*. The *Toledan Tables* were written by the Andalusian astronomer al-Zarqālī (or Azarquiel) in the 11th century. The original Arabic has been lost, but various Latin versions of both the tables and the explanatory text are extant in more than 100 manuscripts scattered all over Western Europe. These manuscripts contain several tables from al-Khwārizmī's original *zīj*, some of which are not found in al-Majrīṭī's recension. The *Toledan Tables* were described in Zinner 1935 and Millás Vallicrosa 1943-1950, pp. 22-71, and were extensively analysed by Toomer (1968). The explanatory text from a group of manuscripts was published by F.S. Pedersen (1987), who is currently preparing a complete edition of the tables.

A commentary on al-Khwārizmī's *zīj* by al-Farghānī, mentioned by al-Bīrūnī and Ibn al-Muthannā, is non-extant. Some of the tables in the only extant manuscript of al-Battānī's *Ṣābi' Zīj* (Escorial árabe 908) are explicitly attributed to Maslama al-Majrīṭī and can thus be used to identify additions by al-Majrīṭī in the Latin translation of al-Khwārizmī's *zīj* (cf. Nallino 1899-1907, vol. 2, pp. 300 ff.).

Valuable information concerning the transmission of Indian and Persian astronomical knowledge to Baghdad in the 8th century can be found in *Kitāb ʿilal al-zījāt* (*The book of the reasons behind astronomical tables*) by ʿAlī ibn Sulaymān al-Hāshimī ("al-Hāshimī" in the bibliography). This information was explored by Pingree in his publications 1968a, 1968b and 1970.

The most important secondary sources dealing with al-Khwārizmī's *Sindhind Zīj* have been mentioned above. Many articles have appeared about particular tables in the *zīj*; these are listed in the bibliography and will be referred to in my survey of previous results about the mathematical structure and origin of the tables in al-Majrīṭī's recension in the following section.

4. *Survey of previous results about al-Khwārizmī's astronomical tables*

In this Section I summarize the most important previous results concerning the mathematical structure and origin of the tables in al-Majrīṭī's recension of al-Khwārizmī's *Sindhind Zīj*. For every table or group of tables references are given to the table numbers in Suter's edition published in 1914 (for tables displaying multiple functions, the respective

columns are indicated by 1°, 2°, etc.); furthermore, to the relevant page numbers in Neugebauer's translation and commentary (1962) and in Goldstein's edition of Ibn al-Muthannā's commentary (1967). In these publications the reader can also find complete technical descriptions of the functions tabulated in al-Khwārizmī's *zīj*. References to other secondary sources will only be given for results that cannot be found in one of the three above-mentioned works. The tables are listed in the order in which they occur in Suter's edition. Note that tables 57b (multiplication of sexagesimal fractional digits) and 116 ("houses, judges and decans") were left out, because they were not mathematically computed. Al-Khwārizmī's tables for timekeeping, the *qibla* and the construction of sundials and astrolabes, which were not part of his *zīj*, are described in King 1983.

CHRONOLOGICAL TABLES
(Suter 1-3a, Neugebauer pp. 82-89, Goldstein pp. 16-25)

Like most Islamic astronomical handbooks, al-Khwārizmī's original *zīj* contained a set of chronological tables similar to the set in the Latin translation of al-Majrīṭī's recension. Al-Majrīṭī made some small modifications to the tables for the *notae* (days of the week of year and month beginnings; cf. Goldstein p. 88 and al-Hāshimī, pp. 231-234). Furthermore, although he maintained the epoch and the year beginning (1 October) of the Byzantine calendar, he moved the intercalary day from the end of February to the end of December (Table 3a).

MEAN MOTIONS
(Suter 4-20, Neugebauer pp. 90-95, Goldstein pp. 26-28 and 190-191)

Al-Khwārizmī's original mean motion tables were calculated for the Persian calendar and the Era Yazdigird. They were based on the Indian mean motion theory, which assumes that at the time of the creation all planets and their apogees and nodes had a mean position equal to 0° Aries. Al-Khwārizmī's original mean motion values were probably given to an accuracy of sexagesimal thirds (cf. Goldstein pp. 28 and 152), and were calculated for the meridian of Ujjain in Central India (in Arabic sources called Arīn).

According to Ṣāʿid al-Andalusī, al-Majrīṭī adapted al-Khwā-rizmī's mean motion tables to the Arabic calendar. The mean motion tables preserved in the Latin translation of al-Khwārizmī's *zīj* are indeed based on the Arabic calendar and are intended for the meridian of Arīn. It can be shown that most of the tables are in agreement with the Indian period relations occurring in Brahmagupta's works (Burckhardt 1961 and Toomer 1964, pp. 207-208; see also Mercier 1987, pp. 90-92).

SOLAR EQUATION
(Suter 21-26 3°, Neugebauer pp. 19-21 and 95-96)

Ibn al-Muthannā gives hardly any information concerning the solar equation table in al-Khwārizmī's original *zīj*. However, there is little doubt that the table in al-Majrīṭī's recension stems from al-Khwārizmī. This table was computed according to the so-called *method of declinations* described by al-Bīrūnī (Kennedy & Muruwwa 1958, p. 118), whereas Indian astronomers used the *method of sines*.[6] Since Ibn al-Qifṭī states that al-Khwārizmī took his planetary equations from "the Persians", it seems plausible that the method of declinations derives from the *Shāh Zīj*. Al-Majrīṭī's maximum solar equation 2°14′ occurs both in the *Khaṇḍakhādyaka* (Neugebauer p. 96) and in the *Shāh Zīj* (Kennedy & Van der Waerden 1963, p. 326). His value 77°55′ for the longitude of the solar apogee is in agreement with the mean motion system used by Brahmagupta (Pingree 1965). The same values for eccentricity and longitude of the apogee were found by Neugebauer (pp. 90-91) to underlie the small table for the mean solar position at the entry of the sun in the zodiacal signs (Suter 4). The solar equation table in al-Majrīṭī's recension was *not* computed by means of linear interpolation between values for multiples of 3¾ °, as suggested by Ibn al-Muthannā (Goldstein pp. 42-43).

LUNAR EQUATION
(Suter 21-26 4°, Neugebauer pp. 21 and 96)

Al-Majrīṭī's recension tabulates only a single lunar equation. Like the solar equation, this table was computed according to the *method of declinations* and has the same maximum value (4°56′) as the *Khaṇḍakhādyaka* (which employs the *method of sines*) and the *Shāh Zīj*. No traces of linear interpolation can be recognized. The equation is probably of

[6] A planetary equation q computed according to the *method of sines* is given by $q(x) = q_{max} \cdot \sin x$, where q_{max} is the maximum equation. An equation computed by the *method of declinations* is given by $q(x) = q_{max} \cdot \delta(x) / \varepsilon$, where δ represents the solar declination for obliquity of the ecliptic ε.

Persian origin.[7]

SOLAR DECLINATION
(Suter 21-26 5°, Neugebauer pp. 96-97, Goldstein pp. 49 and 64-66)

Al-Khwārizmī's original *zīj* contained two tables for the solar declination. In one of these tables al-Khwārizmī followed Ptolemy, although he replaced the obliquity value 23°51′20″ used both in the *Almagest* and in the *Handy Tables* by 23°51′0″. In the other table he followed the Indian tradition by displaying differences between declination and "versed declination" values for multiples of 15° based on the obliquity value 24°. Al-Majrīṭī's recension only contains the Ptolemaic table; the *Toledan Tables* include both the Ptolemaic table (Toomer 1968, pp. 27-28) and, as part of the explanatory text, the Indian values (Millás Vallicrosa 1943-1950, pp. 43-45).

LUNAR LATITUDE
(Suter 21-26 6°, Neugebauer pp. 97-98, Goldstein pp. 89-92 and 211-213)

The lunar latitude table in al-Majrīṭī's recension was computed according to the *method of sines* and has a maximum value of 4°30′. This is in agreement with the commentaries of both Ibn al-Muthannā and Ibn Masrūr (Kennedy & Ukashah 1969, pp. 95-96). The same maximum lunar latitude can be found in Indian sources such as the *Sūryasiddhānta* and the *Khaṇḍakhādyaka* (Sengupta 1934, p. 32) and, according to Ibn Yūnus, in the *Shāh Zīj* (Delambre 1819, pp. 138-139).

PLANETARY EQUATIONS
(Suter 27-56 3°-5°, Neugebauer pp. 22-30 and 98-101, Goldstein pp. 30-45 and 192-198)

Al-Khwārizmī's calculation of the true planetary positions as described by Ibn al-Muthannā is based on Indian methods which were fully explained by Neugebauer (1956, pp. 12-26). The tables and instructions in al-Majrīṭī's recension are in agreement with these methods. The maximum equations agree very well with those from the *Shāh Zīj* as reported by Ibn Hibintā and al-Bīrūnī (Kennedy 1956a, pp. 170-172). The equations of centre were computed according to the *method of sines* using

[7] Note that, for instance, Brahmagupta applies a second correction to the lunar mean motion, which is derived from the solar equation (Sengupta 1934, pp. 21-22).

linear interpolation within intervals of 15°.[8] The equations of anomaly correspond fairly well to the simple eccentric model and are thus approximately given by $\tan q(x) = e \cdot \sin x / (60 + e \cdot \cos x)$, where q is the equation and e the eccentricity. The constant longitudes of the planetary apogees implicit in the column for the *modified apogee* and confirmed by the explanatory text and by Ibn al-Muthannā agree with values calculated from the *Khaṇḍakhādyaka* (Toomer 1964, p. 207).

PLANETARY STATIONS
(Suter 27-56 6°, Neugebauer pp. 30-31 and 101, Goldstein pp. 45-49 and 198)

Both the theory and the tables for the planetary stations in al-Majrīṭī's recension are Ptolemaic. Ibn al-Muthannā confirms the presence of the tables among the planetary equation tables in al-Khwārizmī's original *zīj*. The tabular values are close to those in the *Handy Tables*, but not always identical with them. The same tables for the planetary stations occur in the *Toledan Tables* (Toomer 1968, p. 60).

PLANETARY LATITUDES
(Suter 27-56 7°-8°, Neugebauer pp. 34-41 and 101-103, Goldstein pp. 92-94 and 213-215)

Al-Khwārizmī's rules for the determination of the planetary latitudes given in the commentaries of Ibn Masrūr and Ibn al-Muthannā and in al-Majrīṭī's recension are of Indian origin. The maximum latitudes mentioned in the commentaries are the same as those in al-Majrīṭī's tables and occur in Indian sources like the *Sūryasiddhānta* and the *Khaṇḍakhādyaka*. The second latitude tables (column 8) were computed according to the *method of sines* and are accurate to seconds. The first latitude tables (column 7) are not in full agreement with the Indian rules. Toomer (1964, pp. 205-206) suggested that this could be the result of an error made by al-Majrīṭī when he replaced the value 150 for the radius of the base circle by 60 (cf. the section about the sine below). However, Kennedy & Ukashah (1969) showed that the tables agree with the incorrect explanation of the Indian rules presented in the commentaries of Ibn Masrūr and Ibn al-Muthannā. The constant longitudes of the planetary nodes, given in the tabular headings, agree with calculations based on the *Khaṇḍakhādyaka* (Toomer 1964, p. 207). Al-Majrīṭī's planetary latitude

[8] In the equation of centre for Mars the use of two additional independently calculated values for arguments 82½° and 97½° can be recognized.

tables are also present in the *Toledan Tables* (Toomer 1968, pp. 69-70).

LUNAR VISIBILITY
(Suter 57a, Neugebauer pp. 42-44 and 103, Goldstein pp. 96-104 and 218-225)

The presence of a table for lunar crescent visibility in al-Khwārizmī's original *zīj* cannot be ascertained from the commentaries by Ibn al-Muthannā and Ibn Masrūr. However, a table attributed to al-Khwārizmī can be found in various sources (King 1987, pp. 189-192). This table can be shown to be based on the Indian visibility criterion with obliquity of the ecliptic 23°51′ and geographical latitude 33°. The different table in al-Majrīṭī's recension was studied by Kennedy & Janjanian (1965) and by King (1987, pp. 192-197). A systematic analysis by Hogendijk (1988, pp. 32-35) led to the conclusion that the table was based on the Indian visibility criterion and either obliquity 23°35′ and latitude 41°35′ or obliquity 23°51′ and latitude 41°10′.

SINE
(Suter 58-58a, Neugebauer p. 104, Goldstein pp. 49-62)

Al-Khwārizmī's original *zīj* contained sine and versed sine values for so-called *kardajas* ("sections", multiples of 15 degrees), which were computed for a radius of the base circle equal to 150′. Such values derive from Indian sources (see, for example, the *Khaṇḍakhādyaka*, Sengupta 1934, p. 32) and are also present in the explanatory text of the *Toledan Tables* (Millás Vallicrosa 1943-1950, pp. 43-44). According to Ibn al-Muthannā's commentary, the intermediate values for integer degrees had to be filled in by means of interpolation. One possible way in which al-Khwārizmī could have done this was discovered by Hogendijk (1991). He found that a table for a function called *sine of the hours*, which follows al-Khwārizmī's treatise on the astrolabe in a manuscript in Berlin, is based on the Indian sine values for *kardajas* and a special type of linear interpolation.

The sine table in al-Majrīṭī's recension is based on radius 60 and must therefore be a later addition. Bjørnbo noted that it was computed by halving Ptolemy's chord values and truncating the result after the second sexagesimal fractional digit (1909, pp. 12-13).[9]

RIGHT ASCENSION
(Suter 59-59b, Neugebauer pp. 46-48 and 104-105, Goldstein pp. 69-76 and 202-204)

Al-Khwārizmī's original *zīj* contained a table of the right ascension for every degree of the ecliptic starting with Capricorn, thus following Ptolemy's *Handy Tables*. The table in al-Majrīṭī's recension also starts from Capricorn and, like the solar declination, is based on the obliquity value 23°51′0″. We conclude that it is very probably al-Khwārizmī's original table.

OBLIQUE ASCENSION
(Neugebauer pp. 48-55, Goldstein pp. 76-81 and 204-206)

Neither al-Khwārizmī's original *zīj* nor al-Majrīṭī's recension contain a table for the oblique ascension. Instead it is explained both in Ibn al-Muthannā's commentary and in al-Majrīṭī's recension how to calculate the rising times by means of a right ascension table, a shadow length table for gnomon length $G=12$ units, a table for the *diminutions of the rising times for the entire earth* displaying $R\cdot \tan \delta / G$ (where R is the radius of the base circle and δ the solar declination), and inverse interpolation in a sine table for radius R. These rules are of Indian origin and can also be found in the *Toledan Tables*. Of the required tables al-Majrīṭī's recension contains the right ascension (for al-Khwārizmī's obliquity value 23°51′0″), the shadow length (for $G=12$), and the sine

[9] In my opinion the sine table in al-Majrīṭī's recension of the *Sindhind Zīj* is different from the sine table for radius 60 in the *Toledan Tables*: the number of differences between the two tables which cannot be explained as scribal errors is large enough to make it plausible that the tables were calculated independently (cf. Toomer 1968, p. 29). The *Toledan Tables* also contain a sine table for radius 150 (Toomer 1968, p. 27), which is practically identical to the table in a Latin manuscript with tables for Newminster (England), which was published in Neugebauer & Schmidt 1952, pp. 226-227. From the fact that nearly all values in this table end in 0, 2, 5 or 7, it can be concluded that it was computed from a sine table for radius 60 by multiplying by 2½, possibly in order to construct a set of tables for determining oblique ascensions based on al-Khwārizmī's parameter values (see below). The underlying sine table for radius 60 is different from the table in al-Majrīṭī's recension.

(for $R=60$ instead of al-Khwārizmī's $R=150$), but omits the table for the diminutions. In the *Toledan Tables* we find a table for $R\cdot \tan \delta / G$, which was shown by Lesley (1957, p. 125-127) to be based on $R=150$, $G=12$ and obliquity 23°51′0″.[10] In Ibn al-Muthannā's commentary three values of al-Khwārizmī's table for $R\cdot \tan \delta / G$ are mentioned (Goldstein p. 80; Millás Vendrell 1963, p. 145). Since the table in the *Toledan Tables* displays the same values (disregarding a couple of scribal mistakes), it is probably al-Khwārizmī's original table.

SHADOW LENGTH (cotangent)
(Suter 60, Neugebauer p. 105, Goldstein pp. 87-89)

From Ibn al-Muthannā's commentary it becomes clear that the calculation of the length of the shadow cast by a gnomon is extensively described in al-Khwārizmī's original *zīj*. However, no mention is made of a table for this function. Ibn al-Muthannā states that al-Khwārizmī took the gnomon length equal to 12 units, in agreement with the cotangent table in al-Majrīṭī's recension. Since many Islamic *zīj*es contained a cotangent table for gnomon length 12, it is nevertheless possible that the table is a later addition. In my opinion, the cotangent values in al-Majrīṭī's recension were calculated independently from those in al-Battānī's *zīj* and the *Toledan Tables*.

TRUE SOLAR AND LUNAR MOTION
(Suter 61-66, Neugebauer pp. 57-63 and 105-107, Goldstein pp. 94-96, 104-109, 216-217 and 226-230.)

Suter (p. 90) showed that al-Majrīṭī's table for the true solar and lunar motion and the apparent radii of the sun, moon and shadow agree with the rules given in the *Khaṇḍakhādyaka* and Ibn al-Muthannā's commentary.

EQUATION OF TIME
(Suter 67-68, Neugebauer pp. 63-65 and 107-108)

In Ibn al-Muthannā's commentary no mention is made of the equation of time. Al-Hāshimī presents a Ptolemaic description of the calculation of the equation of time and states that the same method is used

[10] The same table is found in the Latin manuscript with tables for Newminster mentioned in footnote 9; see Neugebauer & Schmidt 1952, p. 226.

in the *Shāh Zīj* and in the *zījes* by al-Khwārizmī and Abū Maᶜshar (al-Hāshimī, pp. 156-157 and 279). He does not give parameter values or other details of the method of computation and does not mention tables for the equation of time in the above-mentioned works.

Al-Majrīṭī's recension of al-Khwārizmī's *zīj* contains a table for the equation of time with values to seconds of an hour for every degree of solar longitude. From the instructions for the use of this table (Suter p. 25; Neugebauer pp. 61-62) it follows that the argument of the table is the true solar longitude and that the equation of time values must always be added to mean solar time to obtain true solar time. The equation of time as tabulated in al-Majrīṭī's recension is typically Ptolemaic; Indian astronomers only corrected for the solar velocity component (cf. Section 5) and thus obtained a sine-wave instead of a function with four local extreme values (cf. Figure 3). In Section 6 of this article the mathematical structure and the underlying parameter values of the table for the equation of time in al-Majrīṭī's recension will be determined.

MEAN OPPOSITIONS AND CONJUNCTIONS

(Suter 69-72, Neugebauer pp. 108-115, Goldstein pp. 94 and 216)

The tables for mean conjunctions and oppositions in al-Majrīṭī's recension were computed for a length of the mean synodic month very close to an Indian value reported by al-Bīrūnī. Since the tables are based on the Arabic calendar and are said to be for the geographical longitude of Cordoba, they were probably modified by al-Majrīṭī. The difference in geographical longitude between the tables for mean oppositions and conjunctions and those for mean motions is approximately 63°. This signifies the first (implicit) occurrence of the so-called "meridian of water", which was used, in particular, by Andalusian and Western-Maghribian geographers and astronomers (Comes 1992-1994, pp. 43-44). The tables for mean oppositions and conjunctions in the *Toledan Tables* are based on parameter values different from those used in al-Majrīṭī's recension (Toomer 1968, pp. 78-81).

LUNAR ECLIPSES

(Suter 73-76, Neugebauer pp. 66-69 and 116-120, Goldstein pp. 109-120 and 231-235)

The organization of the eclipse tables in al-Majrīṭī's recension is purely Ptolemaic. However, Neugebauer found that only the table for lunar eclipses at apogee could be based on Ptolemaic parameter values; the tables for the remaining three cases are based on the Indian value 4°30′ of the maximum lunar latitude. The lunar eclipse tables in al-Majrīṭī's recension are identical to those in the *Toledan Tables* (Toomer 1968, pp. 91-93).

PARALLAX

(Suter 77-77a, Neugebauer pp. 69-76 and 121-126, Goldstein pp. 121-130 and 236-238)

The parallax tables and explanatory text in al-Majrīṭī's recension derive from al-Khwārizmī's original *zīj*. Kennedy (1956b) showed that the latitude component is in complete agreement with the parallax theory in the *Sūryasiddhānta*. The longitude component contains Indian elements as well (in particular the value 24° for the obliquity of the ecliptic), but was computed using an iterative procedure described by Ḥabash al-Ḥāsib (Baghdad, c. 830).

SOLAR ECLIPSES

(Suter 78, Neugebauer pp. 73-76 and 126-128, Goldstein pp. 120-142 and 236-241)

See above under Lunar Eclipses.

EQUATION OF THE HOUSES

(Suter 79-90, Neugebauer pp. 78 and 128-129, Goldstein pp. 84-86 and 209-210)

Ibn al-Muthannā's commentary describes the method by which the equation of the houses can be computed, but does not mention the presence in al-Khwārizmī's original *zīj* of a table for that purpose. The theory underlying the table in al-Majrīṭī's recension is Ptolemaic (Suter pp. 96-98). The underlying parameter values are 23°35′ for the obliquity of the ecliptic and approximately 38°43′ for the geographical latitude (Toomer 1968, pp. 140-143). The table is thus probably an addition by al-Majrīṭī. The same table can be found in the *Toledan Tables*.

PROJECTION OF THE RAYS

(Suter 91-114, Neugebauer pp. 78-81 and 129-131)

The table for the projection of the rays from al-Khwārizmī's original *zīj* can be found in an astrological work by Ibn Hibintā (Kennedy & Krikorian-Preisler 1972) and in the *Toledan Tables* (Toomer 1968, pp. 147-151). Toomer found that the table was computed for obliquity 23°51′ and the latitude of Baghdad (close to 33°). The table for the projection of the rays in al-Majrīṭī's recension is indicated to be for geographical latitude 38°30′, i.e. probably for Cordoba. We can thus conclude that it is an addition by al-Majrīṭī. Hogendijk (1989) discussed the mathematical structure of both tables for the projection of the rays. He found that al-Majrīṭī's table is based on al-Khwārizmī's obliquity value 23°51′, but that it presents a significant improvement of al-Khwārizmī's method of computation.

EXCESS OF REVOLUTION
(Suter 115, Neugebauer pp. 131-132, Goldstein pp. 143-144 and 242)

The table for the excess of revolution in al-Majrīṭī's recension is based on a sidereal year of 365;15,30,22,30 days.[11] This value occurs in various Indian sources, for instance in the *Brāhmasputasiddhānta*, and is confirmed by Ibn al-Muthannā to have been used by al-Khwārizmī. Note that the *Shāh Zīj* uses the value 365;15,32,30 (Kennedy 1956a, p. 147).

Summary

With regard to their origin the tables in the Latin translation of al-Majrīṭī's recension of al-Khwārizmī's *Sindhind Zīj* can be divided into five groups (the numbers mentioned are the table numbers in Suter's edition published in 1914):

I. Tables deriving from al-Khwārizmī's original *zīj*:
A) *based on Indian methods and / or parameter values:*
1) Mean motion tables: motions and positions (4-20)
2) Lunar latitude (21-26 6°)
3) Planetary equations: structure (27-56 3°-5°)
4) Planetary latitudes (27-56 7°-8°)
5) True solar and lunar motion (61-66)
6) Lunar eclipses: parameter values for eclipses at apogee (73-76)
7) Parallax (77-77a)
8) Solar eclipses: parameter values (78)
9) Excess of revolution (115)
B) *based on Persian methods and / or parameter values:*
1) Solar equation (21-26 3°)
2) Lunar equation (21-26 4°)

[11] Sexagesimal numbers will be written in the conventional way: sexagesimal digits are separated by a comma and the sexagesimal point is represented by a semicolon. For example: 365;15,30 or 6,5;15,30 denotes $365 + 15/60 + 30/60^2$.

3) Planetary equations: parameter values (27-56 3°-5°)
C) *based on Ptolemaic methods and / or parameter values:*
1) Solar declination (21-26 5°)
2) Planetary stations (27-56 6°)
3) Right ascension (59-59b)
4) Lunar eclipses: organization and parameter values for eclipses at perigee (73-76)
5) Solar eclipses: organization (78)

II: Tables modified by al-Majrīṭī:
1) Chronological tables (1-3a)
2) Mean motion tables: epoch (4-20)
3) Mean conjunctions and oppositions (69-72)

III: Tables added or replaced by al-Majrīṭī:
1) Lunar crescent visibility (57a)
2) Sine (58-58a)
3) Cotangent (60)
4) Equation of the houses (79-90)
5) Projection of the rays (91-114)

Al-Khwārizmī's original table for lunar crescent visibility is extant in various sources (see above for references). His sine values for *kardajas* based on radius of the base circle 150 can be found in the *Toledan Tables*; a sine table with arguments 1,2,3,...,90 based on these values was reconstructed by Hogendijk. Al-Khwārizmī's original table for the projection of the rays has come down to us in a work by Ibn Hibintā and in the *Toledan Tables*.

The table for the equation of time (67-68) belongs to one of the groups I-C, II or III. The analysis in Section 6 below will enable us to make a more detailed statement about its origin.

5. *The equation of time*

If we want to measure local time by the solar position (for example, by means of a sundial), we define noon as the time of the daily culmination of the sun. The period of time between two consecutive culminations can then be divided into 24 equal hours. In case the sun were

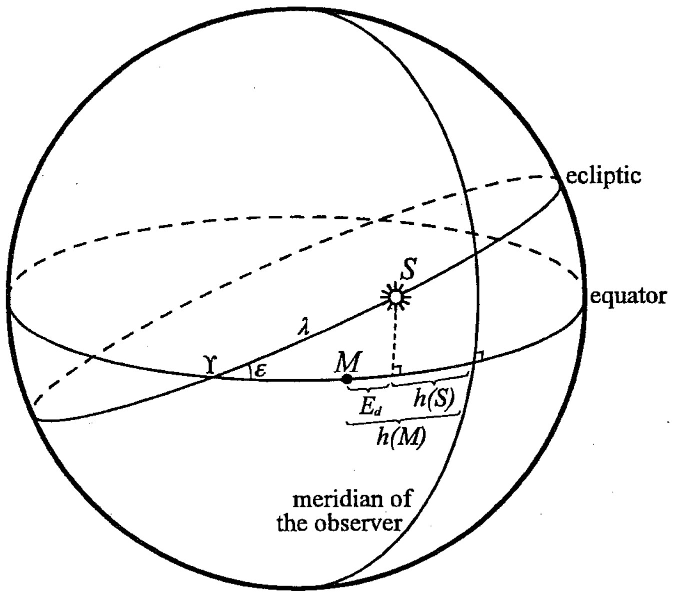

Figure 1 *Graphical explanation of the equation of time.*

positioned in the plane of the equator and moved with a uniform apparent velocity, the arc of the equator which crosses the meridian of the observer between each two consecutive culminations would be the same all through the year, namely 360° plus the daily solar motion. Consequently, each day and each hour would have precisely the same length. The time obtained on the basis of the assumption that the sun moves at a uniform speed in the plane of the equator is called *local mean solar time*; it differs at most a constant amount from the time that we use today. Ancient and medieval astronomers used mean solar time for the calculation of planetary longitudes: they applied corrections to the so-called mean planetary longitudes, which were linear functions of time and could therefore be determined by multiplying the mean solar time elapsed by the average planetary motion.

Since in reality the sun moves in the plane of the ecliptic with a non-uniform velocity, *local true solar time*, which is defined by the daily culmination of the *true* sun, differs a variable quantity from mean solar

time. The difference between true and mean solar time is called the *equation of time* (in Arabic: *taʿdīl al-ayyām bi-layālī-hā*; in Latin: *equatio dierum cum noctibus suis*). It is determined by two factors: the non-uniformity of the solar motion, and the fact that normally an arc of the ecliptic does not cross the meridian of the observer in the same period of time as an equatorial arc of the same length.

We will now define mean and true solar time mathematically and will derive a formula for the equation of time as a function of the solar position.[12] First note that the *hour angle* of a heavenly body X is the spherical angle between the meridian of the observer and the meridian of X, measured in *westward* direction. By S I will denote the true sun; by M the virtual equatorial mean sun, which moves on the equator at a constant speed with the same period as the true sun.

Now mean solar time can be defined as the hour angle $h(M)$ of the equatorial mean sun, true solar time as the hour angle $h(S)$ of the true sun. The equation of time E_d expressed in equatorial degrees is the difference between true and mean solar time (cf. Figure 1, which depicts the heavenly sphere; the earth, located in the centre of the sphere, has not been indicated):

$$E_d = h(S) - h(M). \qquad (1)$$

In order to express E_d as a function of the solar position, we now consider the Ptolemaic eccentric solar model, which was used by most medieval astronomers (see Figure 2).[13] In this model the true sun S moves at a constant speed on a circle with radius 60, which lies in the plane of the ecliptic. The centre C of this circle is a distance e, called the *solar eccentricity*, removed from the Earth E. The sun reaches its largest distance from the Earth at the apogee A, its smallest distance at the

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[12] Extensive descriptions of the equation of time as used by Ptolemy can be found in Neugebauer 1975, vol. 1, pp. 61-68 and Pedersen 1974, pp. 154-158. The methods by which the Islamic astronomers Kūshyār ibn Labbān (10th century) and al-Kāshī (15th century) computed their tables for the equation of time were explained in Kennedy 1988.

[13] More extensive explanations of the Ptolemaic solar model can be found in Neugebauer 1975, vol. 1, pp. 53-61 or Pedersen 1974, pp. 144-154.

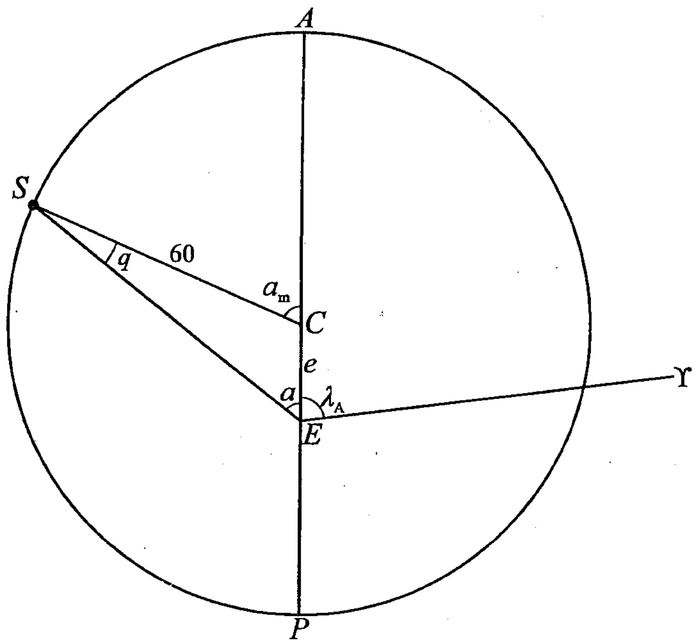

Figure 2 *The Ptolemaic solar model.*

perigee P. Now the *true solar longitude* λ is defined as the angle $\measuredangle$ ϒES, measured in eastward direction from the vernal point ϒ, under which the sun is seen from the Earth (in order to keep Figure 2 as clear as possible I have not indicated λ in it). In the following calculations we will also make use of the *true solar anomaly* a, which is defined by the angle $\measuredangle AES$ between the apogee A and the sun S. We have $\lambda = a + \lambda_A$, where λ_A is the longitude of the solar apogee, given by the angle $\measuredangle$ ϒEA.

In order to calculate the true solar position λ, we will apply a trigonometrically computed correction to a linear function of time. For this function we can, for instance, take the *mean solar anomaly* a_m, which is the angle $\measuredangle ACS$ between the apogee and the sun as seen from the centre C of the eccentric solar orbit. Since the sun moves at a uniform speed around C, this function is in fact linear. However, ancient and medieval astronomers usually tabulated a different function, namely the *mean solar longitude* λ_m, which is defined by $\lambda_m = a_m + \lambda_A$ (for the same reason as above, λ_m has not been indicated in Figure 2).

Since the sum of the angles in triangle ECS is now $a + (180° - a_m) + \measuredangle ESC$, it follows that the difference between the true solar anomaly a and the mean solar anomaly a_m (and hence the difference between the true solar longitude λ and the mean solar longitude λ_m) equals the angle $\measuredangle ESC$. This angle is called the *solar equation* and will be denoted by q. It can be determined geometrically as a function of a_m by extending the triangle SCE to a right-angled triangle SXE (not indicated in the figure) and then expressing the sine or the tangent of the angle q in terms of the sides of the extended triangle. This yields:

$$q^{m}(a^{m}) = \arctan(e \cdot \sin a^{m} / (60 + e \cdot \cos a^{m})) \qquad (2)$$

where q_m denotes the solar equation as a function of the mean solar anomaly (the equivalent formula based on an expression for $\sin q_m$ is somewhat more complicated). For a_m between 0° and 180° the solar equation must be subtracted from the mean solar anomaly (or longitude) in order to obtain the true solar anomaly (or longitude); for a_m between 180° and 360° medieval astronomers added the absolute value of the solar equation to the mean solar anomaly (or longitude). Since our formula yields a negative equation for values of a_m between 180° and 360°, we can write in general $a = a_m - q_m(a_m)$ or $\lambda = \lambda_m - q_m(\lambda_m - \lambda_A)$.

The solar equation can also be expressed as a function of the true solar anomaly and will then be denoted by q. By applying the sine law to triangle SCE we find

$$q(a) = \arcsin(e \cdot \sin a / 60), \qquad (3)$$

and we have $a = a_m - q(a)$ or $\lambda = \lambda_m - q(\lambda - \lambda_A)$ for all values of a and λ.

Now we can return to figure 1. We first note that both the mean solar longitude λ_m and the position of the equatorial mean sun M on the equator are linear functions of time. Since the equatorial mean sun has the same period as the true sun (namely a solar year), it follows that at any time the position of the equatorial mean sun M on the equator can be expressed as $\lambda_m + c$ for a certain constant c. Because the right ascension of a heavenly body X is the spherical angle between the meridian through the vernal point ϒ and the meridian of X measured in *eastward* direction, it follows from formula 1 that at any time the equation of time equals the difference in the right ascension of the equatorial mean sun and the true sun. Thus we have:

$$E_d \;=\; \lambda_m \;-\; \alpha(\lambda) \;+\; c,$$

where α denotes the right ascension. For values of λ between 0° and 90° α can be calculated from $\alpha(\lambda) = \arctan(\cos\varepsilon \cdot \tan\lambda)$, where ε denotes the obliquity of the ecliptic. For values between 90° and 360° α follows from the symmetry relations $\alpha(180-\lambda) = 180 - \alpha(\lambda)$ and $\alpha(180+\lambda) = 180 + \alpha(\lambda)$. The constant c determines the synchronization of the true sun and the virtual equatorial mean sun. Since ancient and medieval astronomers often determined this constant in such a way that it depended on the epoch (i.e. the starting point) of their planetary tables, I will call it the epoch constant.

In the above formula the equation of time E_d is expressed in equatorial degrees. However, in Ptolemy's *Handy Tables* and in most medieval astronomical handbooks (including al-Majrīṭī's recension of al-Khwārizmī's *zīj*) the equation of time was tabulated in hours, minutes and possibly seconds. Since 24 hours correspond to one daily rotation of 360° plus the daily solar motion of approximately 0°59′8″, an accurate factor for the conversion from degrees to hours would be (360° + 0°59′8″) / 24 ≈ 15°2′28″ per hour. However, Ptolemy and many medieval astronomers neglected the daily solar motion and used the factor 15°/ hour instead. Thus if E_h denotes the equation of time expressed in hours, we often have $E_h = 1/15 \cdot E_d$. From now on I will write E instead of E_h, since we will only be dealing with the equation of time expressed in hours.

Like the solar equation, the equation of time can be expressed both as a function of the true solar longitude (denoted by $E(\lambda)$) and as a function of the mean solar longitude (denoted by $E_m(\lambda_m)$). From the above we find

$$E(\lambda) \;=\; 1/15 \cdot (\,\lambda + q(\lambda - \lambda_A) - \alpha(\lambda) + c\,) \qquad (4)$$

and

$$E_m(\lambda_m) \;=\; 1/15 \cdot (\,\lambda_m - \alpha(\,\lambda_m - q_m(\lambda_m - \lambda_A)\,) + c\,) \qquad (5)$$

respectively.

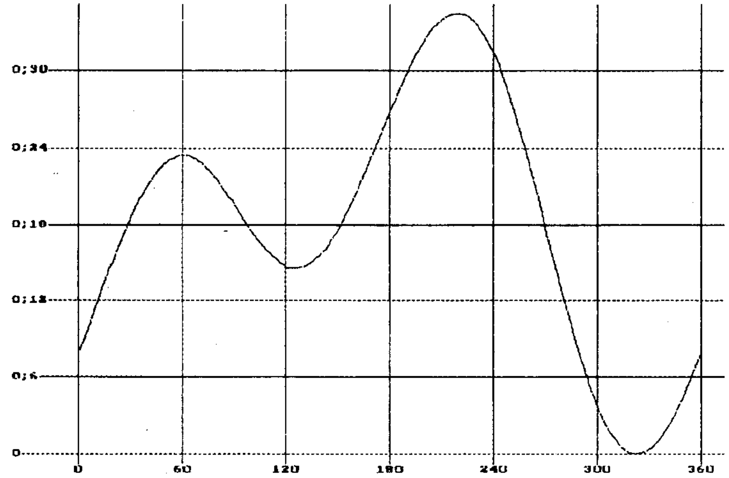

Figure 3 al-Khwārizmī's values for the equation of time (horizontally: the solar longitude in ecliptical degrees; vertically: the equation of time in hours).

We have thus seen that the equation of time is based on five different parameters: the obliquity of the ecliptic, the solar eccentricity, the longitude of the solar apogee, the epoch constant and the conversion factor. Many medieval astronomers included in their handbooks a table for the equation of time, very often without an indication of the underlying parameter values. Furthermore it was not always clear whether the equation of time was tabulated as a function of the mean or the true solar longitude (i.e. whether the *argument* or *independent variable* of the table was the mean or true solar longitude); in both cases a plot of the tabular values would have the general shape shown in Figure 3 (here the solar longitude has been plotted horizontally, the equation of time in hours vertically).

We can conclude that the analysis of a table for the equation of time is often a difficult matter. Until now only very few tables for the equation of time in ancient and medieval sources have been mathematically explained. Kennedy (1988) recomputed the tables for the equation of time found in the *zīj*es of Kūshyār ibn Labbān and al-Kāshī. In a recent article (van Dalen 1994) I have described a number of methods which can be used to analyse tables for the equation of time and have explained in which way Ptolemy computed his table for the equation of time. In this article I will apply similar methods to determine the mathematical structure of al-Khwārizmī's table and will explain the application of the method of least squares in detail.

6. *Analysis of al-Khwārizmī's table for the equation of time*

DESCRIPTION OF THE TABLE.

A complete transcription of the table for the equation of time in the Latin recension of al-Khwārizmī's *Sindhind Zīj* can be found in Suter 1914, pp. 181-182 (Tables 67-68). The table is available in two of the manuscripts mentioned in Section 3 of this article: on folios 80r-80v of Chartres Bibliothèque Publique No. 214 (Suter's manuscript C) and on folios 137r-137v of Oxford Bodleian Library Cod. Auct. F.I. 9 (O). Suter combined the two versions in order to obtain a table which is probably as close as possible to al-Khwārizmī's original table. He found that the two versions have only very few scribal errors in common.

The values of al-Khwārizmī's table for the equation of time are given in minutes and seconds of an hour for every degree of solar longitude starting from Aries. The minimum value is assumed when the solar position is 22° Aquarius and amounts to $0^h0^m0^s$. This implies that the user of the table did not have to distinguish the cases where the tabular values are additive ("positive") and where they are subtractive ("negative"). Apparently the author of the table chose his epoch constant c especially in order to obtain this property (cf. the technical explanation in the previous section).

For the following analysis I have used the values given by Suter (see Tables 4a to 4c at the end of this section), a plot of which is shown in Figure 3. The scarce information about the equation of time in al-Khwārizmī's original *zīj* which can be found in other primary sources has been summarized in the paragraph "Equation of time" of Section 4.

CONVERSION FACTOR.

During a first inspection of al-Khwārizmī's table for the equation of time it can be noted that practically all tabular values are multiples of four seconds. The only exceptions occur in the neighbourhood of the (local and global) minimum and maximum values of the table. It seems

reasonable to assume that also in those regions the values were originally multiples of four seconds, but that they were adjusted in order to avoid obvious jumps in the tabular values.

The presence of mere multiples of 4 seconds can be explained from formulas 4 and 5 for the equation of time: the last step of the calculation is in both cases the division by the conversion factor, which was usually taken equal to 15°/ hour, sometimes to 15°2′28″/ hour. Apparently the author of the table in al-Majrīṭī's recension calculated the equation of time to an accuracy of equatorial minutes, i.e. his values for

$$\lambda + q(\lambda - \lambda_A) - \alpha(\lambda) + c$$

(or $\lambda_m - \alpha(\lambda_m - q_m(\lambda_m - \lambda_A)) + c$ in case the argument of the table was the mean solar longitude) were all multiples of 60 seconds. By dividing by the conversion factor 15 he then obtained tabular values for the equation of time expressed in hours which were all multiples of 4 seconds.

INDEPENDENT VARIABLE.

From the explanatory text in al-Majrīṭī's recension of al-Khwārizmī's *zīj* (Suter 1914, p. 25; Neugebauer 1962, pp. 61-62) it follows that the independent variable of the table for the equation of time is the *true* solar position. Furthermore, the equation found must always be *subtracted* from true solar time to obtain mean solar time; thus the tabular values equal true solar time minus mean solar time as in formula 1.

The second fact can easily be verified from the tabular values: whenever we compute the equation of time by subtracting mean solar time from true solar time, the resulting function has, for a solar position running from 0° to 360°, a local maximum, a local minimum, a global maximum, and a global minimum respectively. As can be seen from Figure 3 (or from Figure 32 in Neugebauer 1962, p. 108) this is in fact the case for the table for the equation of time in al-Majrīṭī's recension.[14]

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[14] Ptolemy's table for the equation of time in the *Handy Tables* is an example of a table of which the tabular values must always be *added* to true solar time in order to obtain mean solar time (cf. Neugebauer 1975, vol. 2, pp. 984-986).

There is no easy way to verify the independent variable of a table for the equation of time: both as a function of the true solar longitude and as a function of the mean solar longitude the equation of time has the form shown in Figure 3. We can, however, derive a number of properties of a table for the equation of time as a function of the true solar longitude which do not hold if the independent variable is the mean solar longitude and which could thus be used to investigate which independent variable was used in al-Majrīṭī's table. In particular, using symmetry relations satisfied by the right ascension and the solar equation we can reconstruct these functions from a given table for the equation of time as a function of the true solar longitude.

Reconstruction of the underlying right ascension and solar equation.

As we have seen in the technical explanation in Section 5, the equation of time is built up from two components, the right ascension and the solar equation. Both components satisfy a number of symmetry relations. In particular, we have for the right ascension α

$$\alpha(180-\lambda) = 180 - \alpha(\lambda) \quad \text{and} \quad \alpha(180+\lambda) = 180 + \alpha(\lambda)$$

for every value of λ. These formulae state mathematically that, for instance, the rising time at sphaera recta of Aries, which can be calculated as $\alpha(30) - \alpha(0)$, is equal to that of Virgo ($\alpha(180) - \alpha(150)$) and to that of Libra ($\alpha(210) - \alpha(180)$).

For the solar equation as a function of the true solar longitude we have

$$q(\lambda_A+a) = -q(\lambda_A-a) \quad \text{and} \quad q(\lambda_A+180+a) = -q(\lambda_A+a)$$

for every value of the true solar anomaly a. Here λ_A is the longitude of the solar apogee and $a = \lambda - \lambda_A$, as in the description of the solar model in Section 5. These formulae state mathematically that the absolute value of the solar equation only depends on the distance of the sun from either apogee or perigee, the sign of the equation being different on the two

sides of the line connecting apogee and perigee.[15]

From the symmetry relations satisfied by the right ascension and the solar equation we can derive relations between certain values for the equation of time as a function of the true solar longitude. First note that, for every value of λ, $E(\lambda) = 1/15 \cdot (\lambda + q(\lambda-\lambda_A) - \alpha(\lambda) + c)$ (formula 4) and

$$\begin{aligned} E(180+\lambda) &= 1/15 \cdot (180 + \lambda + q(180+\lambda-\lambda_A) - \alpha(180+\lambda) + c) \\ &= 1/15 \cdot (180 + \lambda - q(\lambda-\lambda_A) - 180 - \alpha(\lambda) + c) \\ &= 1/15 \cdot (\lambda - q(\lambda-\lambda_A) - \alpha(\lambda) + c). \qquad (6) \end{aligned}$$

Therefore, by *adding* two values for the equation of time for arguments 180° apart, we obtain:

$$\begin{aligned} E(\lambda) + E(180+\lambda) &= 1/15 \cdot (\lambda + q(\lambda-\lambda_A) - \alpha(\lambda) + c + \\ &\qquad \lambda - q(\lambda-\lambda_A) - \alpha(\lambda) + c) \\ &= 1/15 \cdot (2\lambda - 2\alpha(\lambda) + 2c), \end{aligned}$$

from which we find:

$$\alpha(\lambda) = \lambda + c - 7\tfrac{1}{2} \cdot (E(\lambda) + E(180+\lambda)). \qquad (7)$$

Thus we can reconstruct the right ascension underlying a given table for the equation of time as a function of the true solar longitude, provided that we know the value of c (or have a good approximation for it).

By *subtracting* two values for the equation of time for arguments 180° apart, we can in a similar way reconstruct the underlying solar equation. We have:

$$\begin{aligned} E(\lambda) - E(180+\lambda) &= 1/15 \cdot (\lambda + q(\lambda-\lambda_A) - \alpha(\lambda) + c \\ &\qquad - \lambda + q(\lambda-\lambda_A) + \alpha(\lambda) - c) \\ &= 1/15 \cdot (2 \cdot q(\lambda-\lambda_A)), \end{aligned}$$

[15] The solar equation as a function of the mean solar longitude only satisfies a symmetry relation similar to the first of the relations above, namely $q_m(\lambda_A+a_m) = -q_m(\lambda_A-a_m)$, where a_m is the mean solar anomaly ($a_m = \lambda_m - \lambda_A$).

leading to

$$q(\lambda-\lambda_A) \;=\; 7\tfrac{1}{2}\cdot(\,E(\lambda)\;-\;E(180+\lambda)\,). \qquad (8)$$

Thus we can reconstruct the solar equation underlying a table for the equation of time as a function of the true solar longitude even if no value of the epoch constant c is available.

Neither the formulae derived here nor similar formulae hold for the equation of time as a function of the mean solar longitude. Since in formula 5 the solar equation q_m occurs "within" the right ascension (in the term $\alpha(\lambda_m - q_m(\lambda_m-\lambda_A))$), the terms will not cancel out so nicely regardless which equation of time values we add or subtract.

Assuming that al-Khwārizmī's table presents the equation of time as a function of the true solar longitude, we can now reconstruct the underlying solar equation using formula 8. I found that the resulting values are close to solar equation values computed for Ptolemy's solar eccentricity 2;30 and a longitude of the apogee in the neighbourhood of 84°40′, which, as far as I know, is not attested. If the independent variable of al-Khwārizmī's table were the mean solar longitude, the reconstructed table would have shown systematic divergences from solar equation tables computed for any values of the eccentricity and the longitude of the apogee. Therefore we can conclude that the independent variable is *not* the mean solar longitude. More evidence for this conclusion could be found by reconstructing the right ascension underlying al-Khwārizmī's table for the equation of time using formula 7. In order to do that, we first have to find an approximation for the epoch constant c.

APPROXIMATION OF THE EPOCH CONSTANT *C*.

The epoch constant c can be approximated from the values of a table for the equation of time as a function of the true solar longitude by once again applying the symmetry relations satisfied by the right ascension and the solar equation. For every value of λ we have

$$\begin{aligned} E(180-\lambda) &= 1/15\cdot(\,180-\lambda+q(180-\lambda-\lambda_A)-\alpha(180-\lambda)+c\,)\\ &= 1/15\cdot(\,180-\lambda-q(-\lambda-\lambda_A)-180+\alpha(\lambda)+c\,)\\ &= 1/15\cdot(\,-\lambda-q(-\lambda-\lambda_A)+\alpha(\lambda)+c\,) \qquad (9)\end{aligned}$$

and

$$\begin{aligned} E(360-\lambda) &= 1/15\cdot(\,360-\lambda+q(360-\lambda-\lambda_A)-\alpha(360-\lambda)+c\,)\\ &= 1/15\cdot(\,360-\lambda+q(\,180+(180-\lambda)-\lambda_A\,)\\ &\qquad -\alpha(\,180+(180-\lambda)\,)+c)\\ &= 1/15\cdot(\,360-\lambda-q(180-\lambda-\lambda_A)\\ &\qquad -180-\alpha(180-\lambda)+c\,)\\ &= 1/15\cdot(\,360-\lambda+q(-\lambda-\lambda_A)-360+\alpha(\lambda)+c\,)\\ &= 1/15\cdot(\,-\lambda+q(-\lambda-\lambda_A)+\alpha(\lambda)+c\,). \qquad (10)\end{aligned}$$

Using formulae 4, 6, 9 and 10, we can now show that for any value of the true solar longitude λ the sum of the four equation of time values for arguments λ, $180-\lambda$, $180+\lambda$ and $360-\lambda$ is constant:

$$\begin{aligned} E(\lambda)+E(180-\lambda)+E(180+\lambda)+E(360-\lambda) &= \\ = 1/15\cdot(\,\lambda+q(\lambda-\lambda_A)-\alpha(\lambda)+c\,)\,+&\\ 1/15\cdot(\,-\lambda+q(180-\lambda-\lambda_A)+\alpha(\lambda)+c\,)\,+&\\ 1/15\cdot(\,\lambda-q(\lambda-\lambda_A)-\alpha(\lambda)+c\,)\,+&\\ 1/15\cdot(\,-\lambda-q(180-\lambda-\lambda_A)+\alpha(\lambda)+c\,)&\\ = 1/15\cdot(4c)&\\ = 4/15\cdot c.&\end{aligned}$$

If we have a total of n values for the equation of time, where n is a multiple of 4 and the corresponding arguments are $360°/n$, $2\cdot 360°/n$, ..., $360°$, we can build $n/4$ groups of four values of which the sum equals $4/15\cdot c$.[16] As a result, the sum of all available values for the equation of time equals $n/4\cdot(4/15\cdot c)$. This implies that

[16] For $\lambda=0°$ and $\lambda=90°$ we obtain only two values. However, we have
$E(0)+E(90)+E(180)+E(270) =$
$= 1/15\cdot(\,q(0-\lambda_A)+c\,)+1/15\cdot(90+q(90-\lambda_A)-90+c)\,+$
$1/15\cdot(\,180-q(0-\lambda_A)+c\,)+1/15\cdot(\,270-q(90-\lambda_A)-270+c\,)$
$= 4/15\cdot c.$

$$c = (15/n) \cdot \sum_{i=1}^{n} E(i{\cdot}360°/n) \qquad (11)$$

Again, this formula does not hold for the equation of time as a function of the mean solar longitude. However, with a large computational effort it can be shown that for the equation of time as a function of the mean solar longitude formula 11 holds at least approximately, i.e. we have

$$c \approx (15/n) \cdot \sum_{i=1}^{n} E_m(i{\cdot}360°/n)$$

where n is again the total number of tabular values.

Neither for the reconstruction of the underlying right ascension and solar equation tables (formulae 7 and 8) nor for the approximation of the epoch constant c (formula 11) will we in practice be able to use the exact values $E(\lambda)$ for the equation of time. Instead we will have to use tabular values $T(\lambda)$ which were rounded to a fixed number of sexagesimal digits. These tabular values contain at least rounding errors, which are relatively small. (For instance, the differences between exact functional values and values rounded to seconds are half a second at most). Furthermore, they could contain relatively large errors like computational errors or scribal mistakes. Nevertheless, in most cases formula 11 (with E replaced by T) will give us a good approximation for c.

In the case of al-Khwārizmī's table for the equation of time the application of formula 11 leads to $c \approx$ 4;30,3. As was shown above, al-Khwārizmī computed $\lambda + q(\lambda - \lambda_A) - \alpha(\lambda) + c$ to an accuracy of minutes. Furthermore, he apparently chose the epoch constant in such a way that the minimum value of his equation of time became zero. It therefore seems natural to assume that his epoch constant also had an accuracy of minutes. In that case, the value he used was probably c = 4;30.

λ	reconstructed right ascension	differences	λ	reconstructed right ascension	differences
0	0;10, 0	0;10, 0	90	89;48,30	−0;11,30
10	9;20, 0	0;10,20	100	100;44,30	−0;10,14
20	18;33,30	0; 8,47	110	111;33, 0	−0; 9, 1
30	27;56,30	0; 6,20	120	122;11, 0	−0; 4,45
40	37;33,30	0; 3,14	130	132;30,30	−0; 1,35
50	47;27, 0	−0; 0,55	140	142;31,45	0; 2, 1
60	57;38,30	−0; 5,45	150	152;15,30	0; 5,40
70	68; 9,30	−0; 8,29	160	161;43, 0	0; 7,43
80	78;55,30	0; 9,46	170	171; 0, 0	0; 9,40

Table 1 *Right ascension reconstructed from al-Khwārizmī's table for the equation of time under the assumption that the independent variable is the true solar longitude. The 3rd and 6th columns display the differences between the reconstructed values and accurate right ascension values for obliquity 23°51′.*

Assuming that the argument of al-Khwārizmī's table for the equation of time is the true solar longitude and that the epoch constant used was 4;30, we can now reconstruct the underlying right ascension using formula 7. A selection of the resulting values is shown in Table 1 together with the differences between these values and values recomputed for al-Khwārizmī's obliquity value 23°51′. It turns out that the agreement is very bad indeed. Elementary properties of the right ascension such as $\alpha(0)=0$ and $\alpha(90)=90$ are not satisfied by the reconstructed values and we find differences up to 12′ which display an obvious pattern[17] and hence are probably caused by a systematic error in our reconstruction (cf. the explanation of the method of least squares below). It can be checked that this error does not lie in our values of the epoch constant and the obliquity of the ecliptic: for no values of these parameters will the pattern in the differences in Table 1 disappear. We must therefore conclude that the argument of al-Khwārizmī's table is not the true solar longitude.

Since we concluded from the good agreement of the reconstructed solar equation with recomputed values that the argument of

[17]The differences show a clear sine-wave pattern. They are practically zero for $\lambda \approx 45°$ and $\lambda \approx 135°$, reach a maximum of approximately 11′ in the neighbourhood of 0° and 180° and a minimum of approximately −12′ around 90°.

al-Khwārizmī's table for the equation of time is not the mean solar longitude either, we have to consider the possibility that the equation of time was tabulated by methods different from those presented in Section 5. A powerful mathematical tool which can be used to determine multiple unknown parameter values from an astronomical table and to find more information about the tabulated function, is the *method of least squares*. In the following pages this method will be explained in detail.

METHOD OF LEAST SQUARES.

The use of the method of least squares for the determination of the parameter values underlying a given astronomical table will be illustrated by means of Table 2. The 1st column of this table contains arguments λ of a table for the equation of time, the 2nd column tabular values $T(\lambda)$, in this case taken from al-Majrīṭī's recension of al-Khwārizmī's *zīj*. The 3rd column contains equation of time values $E(\lambda)$ computed for a historically plausible set of parameter values, namely obliquity 23°51′, solar eccentricity 2;20 (corresponding to al-Khwārizmī's maximum equation 2°14′), solar apogee 77°55′ (as given by al-Majrīṭī), epoch constant 4°30′ (found above) and conversion factor 15, under the assumption that the independent variable is the true solar longitude. The 4th column contains the differences $T(\lambda)-E(\lambda)$ between al-Majrīṭī's tabular values and our computation, the 5th column the squares of these differences. The sum of the squares (taken over all tabular values present in al-Majrīṭī's recension) is indicated at the end of the table.

From the 4th column of Table 2 it can be seen as follows that either our assumption that al-Khwārizmī's table has the true solar longitude as its independent variable or the chosen parameter values are incorrect. Normally, when we recompute a medieval astronomical table using the correct formula and parameter values, we find differences which have more or less random values and a maximum size of at most a couple of units of the final sexagesimal position. An example of such randomly distributed differences is shown in Figure 4, where the solar position has been plotted horizontally and the differences (each indicated by a dot) vertically. In the 4th column of Table 2 we not only find differences up to 100 units (namely 1′44″), but in a plot of these differences (Figure 5) we can also clearly recognize a non-random pattern, which has more or less the shape of the equation of time itself (cf. Figure 3). The presence of such patterns in the differences usually points to the use of an

arg. λ	$T(\lambda)$ (al-Majrīṭī)	$E(\lambda)$ (computed)	differences $T(\lambda)-E(\lambda)$	squares of the differences
10	0;11,28	0;13, 5,40,24, 1	−0; 1,37,40,24, 1	0; 0, 2,39, 0, 5
20	0;15, 8	0;16,47,55,48,54	−0; 1,39,55,48,54	0; 0, 2,46,26, 3
30	0;18,28	0;20, 2,21,52,44	−0; 1,34,21,52,44	0; 0, 2,28,24,41
40	0;21, 4	0;22,30,18,57, 8	−0; 1,26,18,57, 8	0; 0, 2, 4,10,26
50	0;22,48	0;23,57,55,18,24	−0; 1, 9,55,18,24	0; 0, 1,21,29, 3
60	0;23,28	0;24,18,27,17,28	−0; 0,50,27,17,28	0; 0, 0,42,25,42
70	0;23, 0	0;23,34,23,43,45	−0; 0,34,23,43,45	0; 0, 0,19,43, 3
80	0;21,32	0;21,58,21,35,23	−0; 0,26,21,35,23	0; 0, 0,11,34,50
90	0;19,40	0;19,51,56,41,35	−0; 0,11,56,41,35	0; 0, 0, 2,22,41
100	0;17,32	0;17,42, 7,59,49	−0; 0,10, 7,59,49	0; 0, 0, 1,42,41
110	0;15,52	0;15,56, 0,34,33	−0; 0, 4, 0,34,33	0; 0, 0, 0,16, 5
120	0;14,44	0;14,55,28,26, 2	−0; 0,11,28,26, 2	0; 0, 0, 2,11,39
130	0;14,44	0;14,53,38,17,34	−0; 0, 9,38,17,34	0; 0, 0, 1,32,54
140	0;15,44	0;15,53,39,28,19	−0; 0, 9,39,28,19	0; 0, 0, 1,33,16
150	0;17,36	0;17,49,38,24,14	−0; 0,13,38,24,14	0; 0, 0, 3, 6, 3
160	0;20,24	0;20,28,41,33,54	−0; 0, 4,41,33,54	0; 0, 0, 0,22, 1
170	0;23,32	0;23,33,13,53,43	−0; 0, 1,13,53,43	0; 0, 0, 0, 1,31
180	0;26,52	0;26,43, 2,19, 9	0; 0, 8,57,40,51	0; 0, 0, 1,20,18
190	0;29,52	0;29,36,56,49,11	0; 0,15, 3,10,49	0; 0, 0, 3,46,36
200	0;32,24	0;31,54,16,28, 2	0; 0,29,43,31,58	0; 0, 0,14,43,36
210	0;34, 0	0;33,16,14,32,26	0; 0,43,45,27,34	0; 0, 0,31,54,44
220	0;34,28	0;33,27,36,53,53	0; 1, 0,23, 6, 7	0; 0, 1, 0,46,21
230	0;33,36	0;32,18,41, 6,52	0; 1,17,18,53, 8	0; 0, 1,39,37,34
240	0;31,24	0;29,47,28,59,23	0; 1,36,31, 0,37	0; 0, 2,35,15,30
250	0;27,44	0;26, 1,42,15, 8	0; 1,42,17,44,52	0; 0, 2,54,24,26
260	0;23, 4	0;21,19,28,46,11	0; 1,44,31,13,49	0; 0, 3, 2, 4,32
270	0;17,52	0;16, 8, 3,18,25	0; 1,43,56,41,35	0; 0, 3, 0, 4,32
280	0;12,32	0;11, 0, 1,38,37	0; 1,31,58,21,23	0; 0, 2,20,58,58
290	0; 7,44	0; 6,27,53,26,34	0; 1,16, 6,33,26	0; 0, 1,36,32,37
300	0; 3,48	0; 2,58,35,17, 8	0; 0,49,24,42,52	0; 0, 0,40,41,32
310	0; 1,12	0; 0,49,45,17,10	0; 0,22,14,42,50	0; 0, 0, 8,14,51
320	0; 0, 2	0; 0, 8,24,40,40	−0; 0, 6,24,40,40	0; 0, 0, 0,41, 6
330	0; 0,20	0; 0,51,45,10,37	−0; 0,31,45,10,37	0; 0, 0,16,48,15
340	0; 1,52	0; 2,49, 6, 9,10	−0; 0,57, 6, 9,10	0; 0, 0,54,20,42
350	0; 4,28	0; 5,44, 8,53, 6	−0; 1,16, 8,53, 6	0; 0, 1,36,38,32
360	0; 7,48	0; 9,16,57,40,51	−0; 1,28,57,40,51	0; 0, 2,11,54, 7
SUM OF THE SQUARES OF THE DIFFERENCES:				0; 6,18,43,18, 0

Table 2 *Illustration of the method of least squares*

incorrect formula or incorrect parameter values for the computation.

In order to find the values of the underlying parameters which yield the best agreement with the given table for the equation of time, we can now use the *method of least squares*. The 5th column of table 2 contains the squares of the differences in the 4th column; in the bottom row we find the sum of these squares over all tabular values present in al-Majrīṭī's recension (of these values only every tenth is displayed in Table 2). If we use different sets of parameter values for the computation in the 3rd column, then the differences in the 4th column, the squares of the differences in the 5th column, and the sum of the squares will all be different. *According to the method of least squares, the parameter values are determined in such a way that the sum of the squares of the differences between al-Majrīṭī's table and the computed table is as small as possible.* Expressed mathematically, the parameter values are determined by minimizing the sum

$$\sum_{\lambda} \; (\, T(\lambda) - E(\lambda)\,)^2,$$

which is taken over all values of λ for which tabular values are available. Since squares are positive, the sum of the squares of the differences can only be small if the absolute value of all differences is small, i.e. if all computed values are close to the given tabular values.

Instead of the sum of the squares of the differences we will mostly use the so-called *standard deviation* of the differences. The standard deviation is calculated by dividing the sum of the squares of the differences by the total number of tabular values and taking the square root of the quotient, i.e. the standard deviation is the square root of the average squared difference.[18] The standard deviation is a popular measure for the size of the differences between any two sets of comparable values. In our example in Table 2 the mean square of the differences is approximately 0;0,1,3,7,13 (namely 0;6,18,43,18 / 360), and the standard deviation 0;1,1,32,25. We will see below that if we recompute a table with values to seconds using the correct formula and parameter values, the standard deviation of the differences between tabular values and computed values is approximately 0;0,0,17. Thus for our recomputation of al-Majrīṭī's table the standard deviation of the

[18]For statistical purposes the standard deviation is usually calculated by dividing the sum of the squares of the differences by $n-1$, where n is the total number of tabular values, and taking the square root. In this way the standard deviation yields a better approximation to one of the parameters describing the statistical properties of the differences.

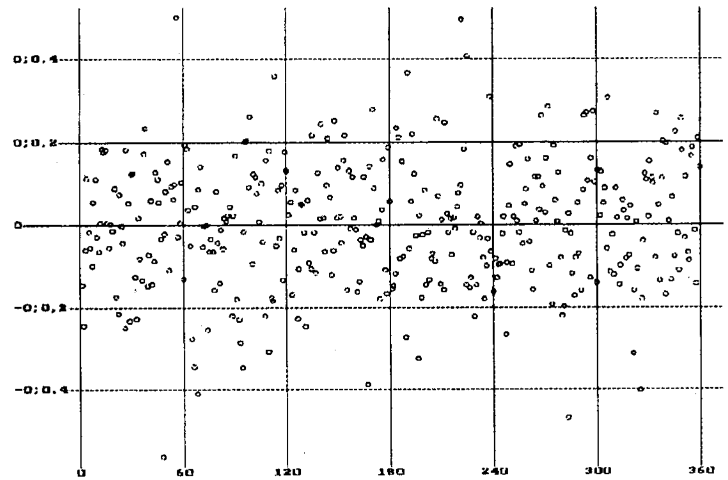

Figure 4 Random differences between equation of time values to seconds and recomputed values (horizontally: the solar longitude; vertically: the differences in hours).

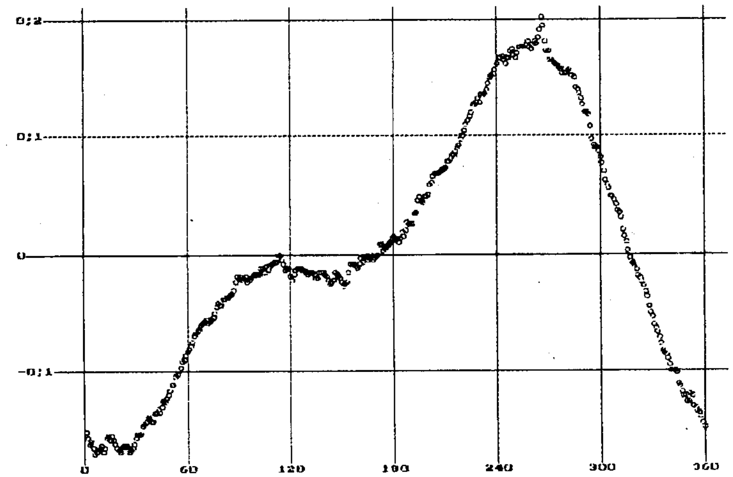

Figure 5 Differences between al-Khwārizmī's values for the equation of time and our recomputation in Table 2.

differences is more than 200 times larger than for a correct recomputation.

It is a complicated numerical problem to determine the parameter values for which the sum of the squares of the differences between a given historical table and a computed table (and hence the standard deviation of these differences) is as small as possible. Usually one has to use an iterative method, which starts with plausible parameter values (such as the ones that we used for our computation in Table 2) and then computes from these values other values for which the sum of the squares of the differences is smaller. Such a computation usually involves the differences between the given table and the table computed for the initial parameter values and the so-called *derivative* of the tabulated function, which supplies information about the speed at which the sum of the squares of the differences changes if the individual parameter values are changed. After a number of repetitions of this procedure (usually only three or four) we obtain a very good approximation to the parameter values for which the sum of the squares of the differences between the given table and a computed table is as small as possible. The values found will be called the *least squares estimates* of the parameters underlying the given table.

In my computer program TA (Table-Analysis) the method of least squares can be applied to determine the underlying parameter values of most of the standard Ptolemaic astronomical tables. The iterative process used by the program is the so-called *method of Gauss-Newton*, which turns out to give very good results for our purpose. The user of TA need not know any details of the iterative process; he merely indicates which parameter values he wants to estimate from which table. However, the interpretation of the results of the method of least squares is non-trivial and will be discussed in the following section.

INTERPRETATION OF THE RESULTS OF THE METHOD OF LEAST SQUARES.

The method of least squares produces accurate approximations to the unknown parameter values of a given astronomical table provided that the *correct underlying function* is used. This implies in the first place that we have to know for which function the given table was computed; for instance, a planetary equation could have been determined according to the method of sines or to the method of declinations (cf. footnote 6). In the second place it is sometimes important to know the exact *method of computation* of the table: if the computation involves sources of systematic

error, such as linear interpolation, severe truncation of intermediate results and the use of inaccurate auxiliary tables, the results of the method of least squares could be invalid. In order to decide whether the results are valid the following three criterions should be applied:

1) *The standard deviation of the differences between the given historical table and a table computed for the least squares estimates of the underlying parameters should be reasonably small.*[19] First note that we cannot expect this standard deviation to be equal to zero. Even if we had chosen the correct underlying function and parameter values for our recomputation of al-Khwārizmī's table for the equation of time in Table 2, the tabular values in the 2nd column would have been rounded versions of the exact values in the 3rd column and the differences in the 4th column would have been between −0;0,0,30 and +0;0,0,30. It can be shown statistically that in that case the standard deviation of the differences between the exact and the rounded values is approximately 0;0,0,17.[20]

As a result the standard deviation of the differences between a given historical table with values to seconds and a table computed for the least squares estimates of the underlying parameters will normally not be smaller than 0;0,0,17. If the standard deviation is much larger than 0;0,0,17, we have to consider the possibility that we have chosen an incorrect underlying function.

2) *The differences between the given historical table and a computation based on the least squares estimates of the underlying parameters should be random and not display obvious patterns.* If the differences display sine-wave or other regular patterns, we can be certain

[19]Bear in mind that this standard deviation is the smallest possible for all sets of parameter values.

[20]The differences between correctly rounded tabular values and exactly calculated functional values can be assumed to have a so-called uniform probability distribution. If the tabular values were calculated to seconds, this implies that all possible digits occur approximately equally often in the third sexagesimal fractional position of the calculated functional values. The expected standard deviation of such uniformly distributed differences can be calculated as approximately 0;0,0,17. If the tabular values were calculated to minutes, the expected standard deviation would be approximately 0;0,17, etc.

that we have used an incorrect underlying function or that we have neglected aspects of the computation of the table which caused systematic errors in the tabular values.[21] If the differences seem to be random, we have probably chosen the correct underlying function even if the standard deviation of the differences is large. Examples of differences with obvious patterns can be found in Figures 5, 6 and 7; an example of random differences is shown in Figure 4.

3) *The least squares estimates should be (close to) historically plausible values for the parameters.* In practice there is only a limited number of possibilities for the values of the underlying parameters of a given historical table. These are either values attested in historical sources (like Ptolemy's value 23°51'20" for the obliquity of the ecliptic and al-Battānī's 2;4,45 for the solar eccentricity) or round numbers (like al-Khwārizmī's value 4°30′ for the epoch constant, which we found above). If the least squares estimates are far removed from historically plausible parameter values, we have probably chosen an incorrect underlying function.

CONFIDENCE INTERVALS.

Even if we have chosen the correct underlying function, the least squares estimates of the parameters of a given astronomical table are normally not identical with the parameter values actually used for the computation. Those values are usually round numbers (see above), whereas the least squares estimates are numerically determined quantities which could in principle have any value; for example, 23;34,59,45,18,6 for the obliquity or 2;4,45,17,23,15 for the solar eccentricity.[22] After

[21]Note that before we applied the method of least squares in our example in Table 2 obvious patterns in the differences had two possible causes: an incorrect underlying function or incorrect values of the underlying parameters. Since we now consider the differences between the given table and a computation based on the least squares estimates of the parameters, which were determined in such a way that the differences are minimized, we can be certain that the cause of the patterns is an incorrect underlying function.

[22]The reason that the least squares estimates are not normally equal to the actual historical parameter values is that the tabular values contain rounding and possibly other types of errors. Even if we would use exact functional values for the

applying the method of least squares we thus have to find historically plausible, round numbers in the neighbourhood of the estimates in such a way that the standard deviation of the differences between the given table and a recomputation for the historical values is only slightly larger than the standard deviation for the least squares estimates. The decision how far the historically plausible values can be removed from the least squares estimates can be made on the basis of so-called *95 % confidence intervals* for the underlying parameters. These are statistically determined intervals around the least squares estimates which are expected to contain the parameter values used for the computation in 19 out of 20 cases.

For example, if we find a 95 % confidence interval ⟨ 23;34,57 , 23;35,6 ⟩ for the obliquity of the ecliptic, we can safely conclude that the underlying parameter value is 23°35′, since this is the only historically plausible value in the neighbourhood of the confidence interval. However, if we find a 95 % confidence interval ⟨ 2;4,27 , 2;4,56 ⟩ for the solar eccentricity, our table could be based on either of the attested values 2;4,35,30 (corresponding to a maximum solar equation of 1°59′) and 2;4,45 (corresponding to 1°59′10″).

APPLICATION OF THE METHOD OF LEAST SQUARES TO AL-KHWĀRIZMĪ'S TABLE.

We have already found that the conversion factor underlying al-Khwārizmī's table for the equation of time is 15°/ hour. Furthermore, we expect that the argument of the table is the true solar longitude. Under these assumptions the results of the application of the method of least squares (as displayed by my program TA) are as follows:

EQUATION OF TIME, AL-KHWĀRIZMĪ (SUTER TABLES 67-68)
LEAST SQUARES ESTIMATION FROM THE VALUES FOR ARGUMENTS 1, 2, ..., 360.

FINAL RESULT (AFTER 3 ITERATIONS)

PARAMETER	ESTIMATE	95 % CONFIDENCE INTERVAL
OBLIQUITY	23;50, 6,30,45, 1	⟨ 23;44,58, 0,34,57 , 23;55,13,58,27,47 ⟩
ECCENTRICITY	2;29,50,28,18,53	⟨ 2;28,39,20,50,30 , 2;31, 1,35,47,15 ⟩
APOGEE	84;40,33,21,39,30	⟨ 84;13,20,52,13, 6 , 85; 7,45,51, 5,54 ⟩
EPOCH CONSTANT	4;30, 3, 0, 0, 0	⟨ 4;29,14,56,34, 9 , 4;30,51, 3,25,51 ⟩

STANDARD DEVIATION OF THE DIFFERENCES: 0;0,31,0,51,32

application of the method of least squares, the estimates need not be equal to the actual parameter values because of the internal rounding in our computer.

Although we have found historically plausible values for the obliquity of the ecliptic and the solar eccentricity (Ptolemy's and al-Khwārizmī's value 23;51 for the obliquity and Ptolemy's value 2;30 for the eccentricity lie in the middle of the respective 95 % confidence intervals), we cannot be satisfied with the result. We have seen that all tabular values are multiples of four seconds. Therefore the standard deviation of the differences between al-Khwārizmī's table and a table computed for the least squares estimates would approximately be 4 · 0;0,0,17 = 0;0,1,28 if we had used the correct function and method of computation. The standard deviation found is more than 20 times as large. Since furthermore the differences display a clear sine wave pattern with an amplitude of approximately 45 seconds (Figure 6), we must conclude that we have not used the correct underlying function, i.e. that al-Khwārizmī's table is not an ordinary table for the equation of time as a function of the true solar longitude.

We will therefore perform the least squares estimation for other possibilities of the underlying function. If we assume that the argument of the table is the *mean* solar longitude, the results are as follows:

EQUATION OF TIME, AL-KHWĀRIZMĪ (SUTER TABLES 67-68)
LEAST SQUARES ESTIMATION FROM THE VALUES FOR ARGUMENTS 1, 2, ..., 360.

FINAL RESULT (AFTER 3 ITERATIONS)

PARAMETER	ESTIMATE	95 % CONFIDENCE INTERVAL
OBLIQUITY	23;35,31,17,18,32	⟨ 23;29,12,34,43,30 , 23;41,48,24,45,48 ⟩
ECCENTRICITY	2;36,11,51,25, 0	⟨ 2;34,41,48,17,40 , 2;37,41,54,32,20 ⟩
APOGEE	85;17,30,12,50,45	⟨ 84;47,11,46, 4, 3 , 85;47,48,39,37,27 ⟩
EPOCH CONSTANT	4;30, 3, 0, 0, 0	⟨ 4;29, 4,42,13,34 , 4;31, 1,17,46,26 ⟩

STANDARD DEVIATION OF THE DIFFERENCES: 0;0,37,37,19,59

We now find a completely different plausible value for the obliquity of the ecliptic (the common Islamic value 23;35), but a practically impossible value for the solar eccentricity. Furthermore, the minimum possible standard deviation is again much larger than the value 0;0,1,28 which we expect for the correct underlying function, and the differences between al-Khwārizmī's table and a table computed on the basis of the estimates again display an obvious pattern (this time more complicated than an ordinary sine-wave; see Figure 7). We conclude that the tabulated function is not the equation of time as a function of the mean solar longitude either.

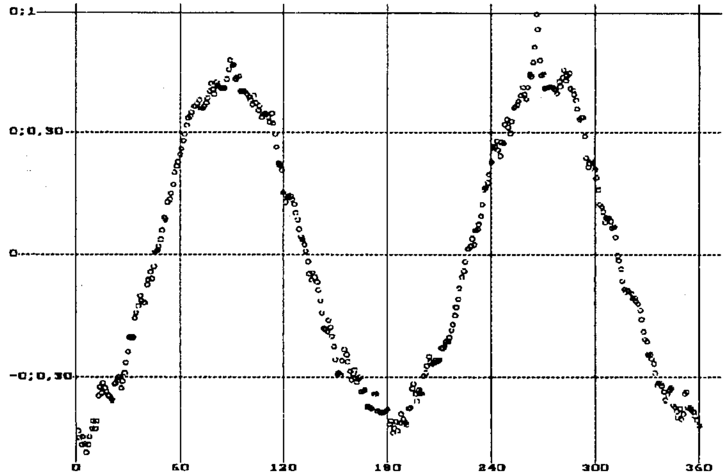

Figure 6 *Differences between al-Khwārizmī's equation of time and the best possible recomputation under the assumption that the argument is the* true *solar longitude.*

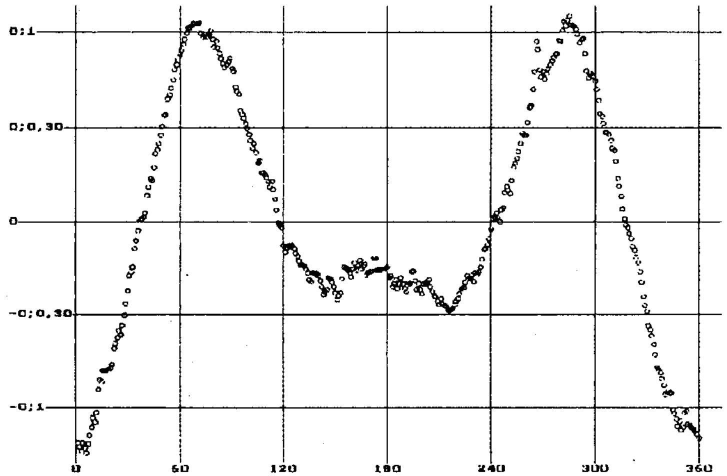

Figure 7 *Differences between al-Khwārizmī's equation of time and the best possible recomputation under the assumption that the argument is the* mean *solar longitude.*

DISPLACED SOLAR EQUATION.

At this point we have to turn to historical sources in order to investigate whether there are still other possible methods of computing tables for the equation of time. In 1988 Kennedy analysed two Islamic tables for the equation of time, namely those in the *Jāmiʿ Zīj* of Kūshyār ibn Labbān (c. 970) and in the *Khāqānī Zīj* of al-Kāshī (c. 1420). Kennedy followed the rules presented in the two *zīj*es and found an excellent agreement between al-Kāshī's table and his own recomputation. However, in the case of Kūshyār's table there remained large systematic differences between the tabular values and recomputed ones.

In my doctoral thesis (1993, pp. 134-141) I investigated Kūshyār's table for the equation of time anew. Like in the present case, an application of the method of least squares did not at once lead to results. Therefore I turned to the text of the *Jāmiʿ Zīj* and found that Kūshyār, who tabulated the equation of time as a function of the mean solar longitude, made use of a so-called *displaced* solar equation. As we have seen in Section 5, the solar equation as determined by Ptolemy and most Islamic astronomers is sometimes additive and sometimes subtractive. This implies that the user of a solar equation table had to decide whether the solar equation must be added to or subtracted from the solar longitude depending on the value of the solar anomaly. Kūshyār avoided this difficulty and made his solar equation $q_m(a_m)$ always additive by subtracting it from 2° and thus obtaining a displaced equation $q_{md}(a_m)$ defined by $q_{md}(a_m) = 2 - q_m(a_m)$ (as before, a_m denotes the mean solar anomaly: $a_m = \lambda_m - \lambda_A$). Kūshyār's approach was not new, since it had, for instance, been used by Ḥabash al-Ḥāsib (c. 830) for his lunar equation tables (Kennedy & Salam 1967, pp. 496-497).[23]

If Kūshyār would add the displaced solar equation $q_{md}(\lambda_m - \lambda_A)$ to the mean solar longitude λ_m, the result would be

$$\lambda_m + q_{md}(\lambda_m - \lambda_A) = \lambda_m + (2 - q_m(\lambda_m - \lambda_A)) = \lambda + 2$$

[23]The manuscript Istanbul Yeni Cami 784/2 of Ḥabash's *zīj* also contains a table for $\lambda_A + q_m(a_m)$, where λ_A is the longitude of the solar apogee and $q_m(a_m)$ the solar equation as a function of the mean solar anomaly (cf. Debarnot 1987, p. 58). From this table the true solar position can be calculated by taking the tabular value for the desired mean solar anomaly and adding it to that anomaly.

λ_m	"ordinary" solar equation	λ_m	displaced solar equation	λ_{ms}	shifted displaced solar equation
356	−0; 8, 2	356	2; 8, 2	356	2; 4, 1
357	−0; 6, 1	357	2; 6, 1	357	2; 2, 1
358	−0; 4, 1	358	2; 4, 1	358	2, 0, 0
359	−0; 2, 1	359	2; 2, 1	359	1;57,59
0	0; 0, 0	0	2, 0, 0	0	1;55,59
1	0; 2, 1	1	1;57,59	1	1;53,59
2	0; 4, 1	2	1;55,59	2	1;51,58
3	0; 6, 1	3	1;53,59	3	1;49,58
4	0; 8, 2	4	1;51,58	4	1;47,58
86	1;58,30	86	0; 1,30	86	0; 1,10
87	1;58,41	87	0; 1,19	87	0; 1, 2
88	1;58,50	88	0; 1,10	88	0; 0,56
89	1;58,58	89	0; 1, 2	89	0; 0,52
90	1;59, 4	90	0; 0,56	90	0; 0,50
91	1;59, 8	91	0; 0,52	91	0; 0,52
92	1;59,10	92	0; 0,50	92	0; 0,57
93	1;59, 8	93	0; 0,52	93	0; 1, 4
94	1;59, 3	94	0; 0,57	94	0; 1,12

Table 3 *The displacement and the shift of Kūshyār's solar equation table*

instead of the desired true solar longitude λ itself (cf. Section 5). Kūshyār therefore replaced λ_m by a *shifted* mean solar longitude λ_{ms} defined by $\lambda_{ms} = \lambda_m - 2$. The addition of the displaced solar equation to the corresponding shifted mean solar longitude then yielded the true solar longitude λ. In order to tabulate the displaced solar equation as a function of the shifted mean solar longitude Kūshyār had to shift all values two degrees backwards, thus tabulating

$$q_{md}(\lambda_{ms} - \lambda_A) = 2 - q_m(\lambda_{ms} - \lambda_A + 2)$$

(cf. Table 3).[24] In that way he could calculate the true solar position

[24]Thus the zeros $q_m(0°)=0;0,0$ and $q_m(180°)=0;0,0$ of Kūshyār's original solar equation lead to displaced solar equation values $q_{md}(-2°) = (q_{md}(358°)=)$ 2;0,0 and $q_{md}(178°) = 2;0,0$. The maximum value $q_m(92°) = 1;59,10$ leads to a minimum $q_{md}(90°) = 0;0,50$ and the minimum value $q_m(268°) = -1;59,10$ to a maximum $q_{md}(266°) = 3;59,10$ (in each case the argument of q_{md} is the shifted mean solar

corresponding to a given shifted mean solar longitude λ_{ms} by adding $q_{md}\,(\lambda_{ms}-\lambda_A)$ to λ_{ms}:

$$\begin{aligned}\lambda_{ms} + q_{md}\,(\lambda_{ms} - \lambda_A) &= (\lambda_m - 2) + (2 - q_m\,(\lambda_{ms} + 2 - \lambda_A)) \\ &= \lambda_m - q_m\,(\lambda_m - \lambda_A) \\ &= \lambda\end{aligned}$$

It now seems natural that also the argument of Kūshyār's table for the equation of time was the shifted mean solar longitude λ_{ms} instead of the ordinary mean solar longitude λ_m. Thus we expect that the tabulated function is:

$$\begin{aligned}E_{ms}(\lambda_{ms}) &= E_m\,(\lambda_{ms}+2) \\ &= 1/15 \cdot (\lambda_{ms} + 2 - \alpha\,(\lambda_{ms} + 2 - q_m\,(\lambda_{ms}+2)\,) + c)\end{aligned}$$

(cf. formula 5 and note that the shifted equation of time for argument λ_{ms} corresponds to the ordinary equation of time for argument λ_m, which is equal to $\lambda_{ms} + 2°$). We thus see that the equation of time as a function of the shifted mean solar longitude can be obtained from the "ordinary" equation of time by shifting all values two degrees backwards.

If we take into account that what Kūshyār calls the "mean solar longitude" is in fact a *shifted* mean solar longitude, we find a good agreement between his table for the equation of time and a recomputation following the rules presented in his *zīj* (van Dalen 1993, pp. 138-139).

The shift in al-Khwārizmī's table for the equation of time.

Different from Kūshyār's table for the equation of time, al-Khwārizmī's table is expected to have the true solar longitude as its independent variable. Although al-Khwārizmī's solar equation is not of the displaced type described above, it could nevertheless be worth while to investigate whether his equation of time values were shifted. For a given shift Δ we define the shifted true solar longitude λ_s by $\lambda_s = \lambda - \Delta$. The shifted equation of time E_s as a function of λ_s is then given by:

$$\begin{aligned}E_s\,(\lambda_s) &= E\,(\lambda_s + \Delta) \\ &= 1/15 \cdot (\lambda_s + \Delta + q\,(\lambda_s + \Delta) - \alpha(\lambda_s + \Delta) + c).\end{aligned}$$

longitude).

Again the resulting function is obtained from the "ordinary" equation of time by shifting all tabular values Δ degrees backwards. However, because of the shift some of the properties of the equation of time as a function of the true solar longitude which we derived from the symmetry relations satisfied by the right ascension and the solar equation (formulas 7, 8 and 11 above) are no longer valid. Formula 11 now holds only approximately, formula 8 yields a shifted solar equation $q(\lambda_s+\Delta) = 7\frac{1}{2} \cdot (\,E_s(\lambda_s) - E_s(180+\lambda_s)\,)$, and instead of formula 7 we obtain:

$$\alpha\,(\lambda_s+\Delta) - (\lambda_s+\Delta) = c - 7\tfrac{1}{2} \cdot (\,E_s(\lambda_s) + E_s(\lambda_s + 180°)\,) \qquad (12)$$

for every value of λ_s. Because $\lambda_s+\Delta$ equals λ and $\alpha(\lambda) - \lambda = 0$ whenever λ is a multiple of 90°, we expect that the right hand side of formula 12 equals 0 whenever λ_s equals a multiple of 90° minus Δ. Since we usually do not have an exact value for c and since the values of λ_s for which the right hand side of formula 12 is precisely equal to zero need not be among the arguments of our table, this property allows us only in exceptional cases to determine the shift.

A more effective method for determining the shift is to regard it as a fifth parameter of the equation of time and to approximate it together with the other underlying parameters using the method of least squares. If we assume that the independent variable of al-Majrīṭī's table is a shifted true solar longitude, the results are as follows:

Equation of Time   al-Khwārizmī (Suter Tables 67-68)
LEAST SQUARES ESTIMATION from the values for arguments 1, 2, ..., 360.

FINAL RESULT (after 3 iterations)

Parameter	Estimate	95 % confidence interval
Obliquity	23;51,51, 2,41,32	⟨23;51,21, 8,10,36 , 23;52,20,56,37,11⟩
Eccentricity	2;29,50,28,18,53	⟨ 2;29,43,33,23,37 , 2;29,57,23,14, 8⟩
Apogee	82;39, 3,53,30,19	⟨82;36, 8,48, 1, 3 , 82;41,58,58,59,35⟩
Epoch constant	4;30, 3, 0, 0, 0	⟨ 4;29,58,19,38,52 , 4;30, 7,40,21, 8⟩
Shift	−2; 1,29,28, 9,11	⟨−2; 2,43,23, 2,39 , −2; 0,15,33,15,43⟩

Standard deviation of the differences: 0;0,3,0,55,44

We first note that the minimum possible standard deviation of the differences between al-Khwārizmī's table and computed values based on the assumption of a shift is much smaller than the standard deviations we obtained before. In fact, the standard deviation found is only twice as large as the value 0;0,1,28 which we expect if we have chosen the correct underlying function for a table of which all values are multiples of four seconds.

Secondly we note that the least squares estimates for the shifted equation of time are close to historically plausible values for all underlying parameters: Ptolemy's and al-Khwārizmī's value 23°51′ for the obliquity of the ecliptic, Ptolemy's value 2;30 for the solar eccentricity, and the value 82°39′ (or possibly 82°40′) for the longitude of the solar apogee. This value was determined from observations made by order of the caliph al-Ma'mūn (c. 830) and was used in the *zījes* of al-Khwārizmī's contemporaries Yaḥyā ibn Abī Manṣūr and Ḥabash al-Ḥāsib. The value 4;30 for the epoch constant is in agreement with what we have found before using formula 11, and the estimated shift is close to −2° (i.e. 2 degrees forwards). The fact that some of the plausible parameter values lie just outside of their 95 % confidence intervals, could point to small systematic errors in the computation of the table. As was mentioned before, possible causes of such errors are linear interpolation in the equation of time itself or in the underlying tables, systematic truncation of intermediate results, etc.

If we recompute al-Khwārizmī's table for the equation of time for the historically plausible parameter values mentioned above, we find that the differences between table and recomputation are generally smaller than 7 seconds and display no obvious global pattern (see Tables 4a to 4c and Figure 8). There are some local patterns in the differences (for example, the small mountains around arguments 6, 65 and 216, and the somewhat larger one around 266°). These could be indications of the small systematic errors indicated above. However, the general pattern of the differences is random enough to conclude that the historically plausible parameter values found above were in fact used for the computation of al-Khwārizmī's table for the equation of time.

The use of the method of least squares has (indirectly) confirmed that al-Khwārizmī's table displays the equation of time as a function of the true solar longitude and that the conversion factor used is 15°/ hour. If we apply the method of least squares for the equation of time as a function of a shifted *mean* solar longitude, we find a minimum possible standard deviation of 19 seconds and differences between table and recomputation

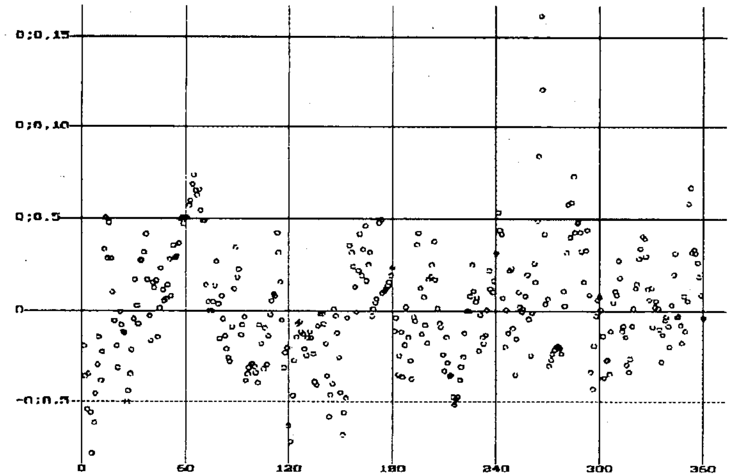

Figure 8 Differences between al-Khwārizmī's values for the equation of time and our final recomputation under the assumption that the tabular values were shifted.

showing obvious sine-wave patterns. If we assume that the conversion factor is 15;2,28 instead of 15, the minimum possible standard deviation of the differences is again 3 seconds and the differences are just as random as in the case of conversion factor 15, but the least squares estimates are further away from historically plausible values.

Assuming that the shift of al-Khwārizmī's table for the equation of time is −2° precisely, we can easily reconstruct the underlying "ordinary" table for the equation of time. From this table we can reconstruct the right ascension and the solar equation according to formulas 7 and 8. It turns out that both underlying tables contain large groups of small errors of the same sign pointing to the presence of some source of small systematic error. I have not been able to determine this source, but it is probably the same one that also caused the above-mentioned local patterns in the differences in Tables 4a to 4c and Figure 8.

λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff
1	8, 8	−2	31	18,48	+2	61	23,28	+5	91	19,28	+2
2	8,28	−4	32	19, 4		62	23,28	+6	92	19,12	−1
3	8,48	−5	33	19,20	−1	63	23,27	+6	93	19, 0	−1
4	9,12	−3	34	19,40	+3	64	23,26	+7	94	18,48	
5	9,32	−6	35	19,56	+3	65	23,24	+7	95	18,32	−4
6	9,52	−8	36	20,12	+3	66	23,20	+7	96	18,20	−3
7	10,16	−6	37	20,28	+4	67	23,16	+6	97	18, 8	−3
8	10,40	−5	38	20,40	+2	68	23,12	+7	98	17,56	−3
9	11, 4	−3	39	20,52		69	23, 6	+5	99	17,44	−3
10	11,28	−1	40	21, 4	−2	70	23, 0	+5	100	17,32	−3
11	11,48	−4	41	21,20	+2	71	22,54	+5	101	17,20	−3
12	12,12	−2	42	21,32	+1	72	22,44	+1	102	17, 8	−4
13	12,40	+3	43	21,44	+2	73	22,36		103	17, 0	−1
14	13, 4	+5	44	21,52	−1	74	22,28		104	16,48	−2
15	13,24	+3	45	22, 4		75	22,20		105	16,36	−3
16	13,48	+5	46	22,16	+2	76	22,12		106	16,28	−1
17	14, 8	+3	47	22,24	+1	77	22, 4	+1	107	16,16	−3
18	14,28	+1	48	22,32	+1	78	21,56	+3	108	16, 8	−1
19	14,48	−1	49	22,40	+1	79	21,44		109	16, 0	
20	15, 8	−2	50	22,48	+1	80	21,32	−2	110	15,52	+1
21	15,28	−3	51	22,56	+3	81	21,24	+1	111	15,44	+1
22	15,52		52	23, 0	+1	82	21,12		112	15,36	+1
23	16,12	−1	53	23, 8	+4	83	21, 0	−1	113	15,32	+4
24	16,32	−1	54	23,12	+3	84	20,48	−2	114	15,24	+3
25	16,52	−1	55	23,16	+3	85	20,36	−3	115	15,16	+2
26	17, 8	−5	56	23,20	+4	86	20,24	−3	116	15, 8	−1
27	17,28	−4	57	23,24	+5	87	20,14	−1	117	15, 0	−3
28	17,48	−3	58	23,26	+5	88	20, 4	+1	118	14,56	−2
29	18, 8	−2	59	23,27	+5	89	19,54	+3	119	14,52	−2
30	18,28		60	23,28	+5	90	19,40	+2	120	14,44	−6

Table 4a *al-Khwārizmī's values for the equation of time (T(λ)) and the differences between al-Khwārizmī's values and our final recomputation (1st part)*

λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff
121	14,40	−7	151	17,48	−7	181	27, 8	−1	211	34, 8	−1
122	14,40	−5	152	18, 4	−6	182	27,28		212	34,12	−3
123	14,40	−3	153	18,20	−5	183	27,44	−4	213	34,16	−4
124	14,40	−1	154	18,40		184	28, 4	−2	214	34,20	−4
125	14,40	−1	155	19, 0	+4	185	28,24	−1	215	34,22	−5
126	14,40	−1	156	19,16	+3	186	28,40	−4	216	34,24	−5
127	14, 0	−1	157	19,32	+2	187	29, 0	−2	217	34,26	−5
128	14,41	−1	158	19,48	+1	188	29,20		218	34,27	−5
129	14,42	−2	159	20, 4		189	29,36	−1	219	34,28	−4
130	14,44	−2	160	20,24	+2	190	29,52	−3	220	34,28	−3
131	14,48	−2	161	20,44	+4	191	30, 8	−4	221	34,27	−3
132	14,52	−1	162	21, 0	+2	192	30,28		222	34,26	−1
133	14,56	−1	163	21,20	+3	193	30,44	−1	223	34,24	
134	15, 0	−2	164	21,40	+5	194	31, 4	+4	224	34,20	
135	15, 4	−4	165	21,56	+2	195	31,20	+4	225	34,16	+1
136	15,10	−4	166	22,16	+3	196	31,32	+1	226	34,12	+3
137	15,20	−1	167	22,36	+3	197	31,44	−1	227	34, 4	+1
138	15,28		168	22,52		198	32, 0	+1	228	33,56	
139	15,36		169	23,12		199	32,12	−1	229	33,48	+1
140	15,44	−1	170	23,32		200	32,24	−2	230	33,36	−2
141	15,52	−2	171	23,52	+1	201	32,40	+2	231	33,28	
142	16, 0	−4	172	24,16	+5	202	32,52	+2	232	33,16	−1
143	16, 8	−6	173	24,36	+5	203	33, 4	+3	233	33, 4	−2
144	16,20	−5	174	24,52	+1	204	33,16	+4	234	32,52	−1
145	16,32	−4	175	25,12	+1	205	33,24	+2	235	32,40	
146	16,48		176	25,32	+1	206	33,32		236	32,28	+2
147	17, 0		177	25,52	+1	207	33,40	−1	237	32,12	+1
148	17,12	−1	178	26,12	+2	208	33,48	−1	238	31,56	+1
149	17,24	−3	179	26,32	+2	209	33,54	−2	239	31,40	+2
150	17,36	−4	180	26,52	+2	210	34, 0	−3	240	31,24	+3

Table 4b *al-Khwārizmī's values for the equation of time (T(λ)) and the differences between al-Khwārizmī's values and our final recomputation (2nd part)*

λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff	λ	T(λ)	diff
241	31, 8	+5	271	17,16	−3	301	3,28		331	0,24	−1
242	30,48	+4	272	16,44	−3	302	3, 8	−1	332	0,32	
243	30,28	+4	273	16,12	−2	303	2,48	−4	333	0,40	+1
244	30, 4	+1	274	15,40	−2	304	2,32	−3	334	0,48	
245	29,40	−2	275	15, 8	−2	305	2,16	−3	335	0,56	−1
246	29,20		276	14,36	−2	306	2, 0	−3	336	1, 4	−3
247	28,56	−1	277	14, 4	−2	307	1,48	−1	337	1,16	−1
248	28,36	+2	278	13,32	−2	308	1,36		338	1,28	−1
249	28,12	+2	279	13, 4	+1	309	1,24	+1	339	1,40	
250	27,44	−1	280	12,32		310	1,12	+1	340	1,52	−1
251	27,16	−4	281	12, 4	+3	311	1, 3	+3	341	2, 4	−2
252	26,52	−2	282	11,36	+6	312	0,52	+2	342	2,20	
253	26,28	+1	283	11, 4	+4	313	0,40	−1	343	2,36	+2
254	26, 0		284	10,36	+6	314	0,32	−1	344	2,52	+3
255	25,32		285	10, 8	+7	315	0,24	−1	345	3, 4	
256	25, 4		286	9,36	+4	316	0,16	−3	346	3,20	
257	24,36	+1	287	9, 8	+5	317	0,10	−3	347	3,36	−1
258	24, 8	+2	288	8,40	+5	318	0, 6	−3	348	3,52	−2
259	23,36		289	8,12	+4	319	0, 4	−1	349	4,12	+1
260	23, 4	−2	290	7,44	+3	320	0, 2		350	4,28	−1
261	22,36		291	7,16	+2	321	0, 1	+1	351	4,48	+1
262	22, 8	+3	292	6,52	+3	322	0, 0	+1	352	5,12	+6
263	21,36	+2	293	6,28	+4	323	0, 1	+3	353	5,32	+7
264	21, 8	+5	294	6, 0	+1	324	0, 2	+3	354	5,48	+3
265	20,40	+8	295	5,32	−3	325	0, 4	+4	355	6, 8	+3
266	20,16	+16	296	5, 8	−4	326	0, 6	+4	356	6,28	+3
267	19,40	+12	297	4,48	−2	327	0, 8	+3	357	6,48	+3
268	19, 0	+4	298	4,28		328	0,10	+1	358	7, 8	+2
269	18,24		299	4, 8	+1	329	0,14	+1	359	7,28	+1
270	17,52	+1	300	3,48	+1	330	0,20	+1	360	7,48	

Table 4c *al-Khwārizmī's values for the equation of time (T(λ)) and the differences between al-Khwārizmī's values and our final recomputation (3rd part)*

7. *Conclusions*

The mathematical analysis of the table for the equation of time in the Latin translation of al-Majrīṭī's recension of al-Khwārizmī's *Sindhind Zīj* has led to the following results:

A. The independent variable of the table is the true solar longitude, in agreement with the explanatory text in the Latin translation of al-Majrīṭī's recension.

B. The factor used for the conversion from equatorial degrees to hours is 15°/ hour. This can be concluded from the fact that practically all tabular values are multiples of 4″ and is confirmed by an application of the method of least squares.

C. The underlying value of the obliquity of the ecliptic is 23°51′. This value also underlies the tables for the solar declination and the right ascension in al-Majrīṭī's recension. It is a rounded version of the value 23°51′20″ used by Ptolemy in the *Almagest* and the *Handy Tables*.

D. The underlying solar equation was computed on the basis of the Ptolemaic solar theory. The value of the eccentricity is Ptolemy's 2;30, which corresponds to a maximum equation of 2°23′. The solar equation table in al-Majrīṭī's recension is of Indian / Persian origin and is based on a maximum equation of 2°14′.

E. The underlying longitude of the solar apogee is 82°39′, the value determined by the group of astronomers working for the caliph al-Ma'mūn (c. 830).[25] Note that neither the Indian value 77°55′ given in al-Majrīṭī's instructions for calculating the true solar longitude, nor the Ptolemaic value 65°30′ were used. It seems natural that Ptolemy's outdated longitude of the solar apogee was replaced with the result of recent observations, but then the same should have been done with the solar eccentricity (the maximum solar equation determined by al-Ma'mūn's astronomers was 1°59′).

F. The underlying value of the epoch constant is 4°30′. As we have seen, the epoch constant was determined in such a way that the

[25]On the basis of the 95 % confidence intervals given above we cannot choose between the value 82°39′, which was used in the *zījes* of Yaḥyā ibn Abī Manṣūr and Ḥabash al-Ḥāsib, and the rounded value 82°40′, which occurs in a table in Ḥabash's *zīj* (see Debarnot 1987, p. 58).

minimum equation of time became zero. Since the minimum occurs for argument 322° (22° Aquarius), corresponding to argument 320° for the unshifted table, we expect c ≈ $\alpha(320) - 320 + q(320-\lambda_A)$ (cf. formula 4). For the parameter values found above this yields c ≈ 4;30,22, which is rounded to 4;30.

G. The values for the equation of time in al-Majrīṭī's recension were shifted forwards 2 degrees, i.e. the actual equation of time value for 0° occurs for argument 2°, the one for 1° for argument 3°, etc. I have not been able to find a satisfactory explanation for this shift. Neugebauer explains how a small shift in the solar longitude is required to make the minimum equation of time equal to 0 (1962, pp. 64-65); however, this shift is smaller than 1′. Furthermore, there is no reason to believe that al-Khwārizmī's table for the equation of time belonged to a set of solar tables based on a displaced equation, such as Kūshyār's. In fact, al-Khwārizmī should have chosen his displacement larger than 2°, since his maximum solar equation is 2°23′.

From the above we can conclude that the table for the equation of time in the Latin translation of al-Majrīṭī's recension of al-Khwārizmī's *Sindhind Zīj* fits into the group of Ptolemaic tables in that work which probably stem from al-Khwārizmī (group I-C in Section 4): it is based on the Ptolemaic values for the obliquity and the solar eccentricity and has a minimum value equal to zero following the table for the equation of time in the *Handy Tables*. The longitude of the solar apogee stems from the astronomers employed by the caliph al-Ma'mūn and was used in the earliest Islamic astronomical handbooks which were mainly based on the Ptolemaic planetary models. Nevertheless, we cannot be certain that the table was computed by al-Khwārizmī, since none of the sources listed in Section 3 mentions a table for the equation of time in al-Khwārizmī's original *zīj*. In any case we can conclude that either the whole table or the underlying parameter values were transmitted from Eastern to Western Islam.

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